ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 205 - Basic Linear Algebra

FINAL EXAMINATION
07.01.2016

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 120 minutes

Question	Grade	Out of
1		20
2		25
3		25
4		20
5		20
Total		110

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Let V be the Euclidean space $\mathbb{R}^3$ with the inner product defined by $< a, b >= a_1b_1 + 2a_2b_2 + 3a_3b_3$ where $a = (a_1,\, a_2,\, a_3,)$ and $b = (b_1,\, b_2,\, b_3)$. Let

$$S = \{v_1 = (2, -1, 1),\ v_2 = (3, 0, 1),\ v_3 = (4, 2, -1)\}.$$

(a) Show that S is a basis for $\mathbb{R}^3$.

(b) Apply Gram-Schmidt process to transform the basis S into an orthogonal basis T.

Question 2. Given the linear transformation $L : \mathbb{R}^5 \to \mathbb{R}^4$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 + 3x_4 - x_5 \\ x_1 + 2x_4 - x_5 \\ 2x_1 - x_3 + 5x_4 - x_5 \\ -x_3 + x_4 \end{bmatrix}$$

(a) Find a basis for $\text{Ker}(L)$.

(b) Find a basis for $\text{Range}(L)$.

(c) Find $\text{nullity}(L)$ and $\text{rank}(L)$.

(d) Is L one-to-one and onto? Justify your answer.

(e) What must be the value of a if $\begin{bmatrix} a \\ 2a \\ 3 \\ -3 \end{bmatrix} \in \text{Range}(L)$?

Question 3.

(a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(i) Compute $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

(ii) Prove that L is invertible.

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(b) Solve the following linear system by Cramer's rule.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 2. \end{aligned}$$

Question 4.

(a) Determine whether $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_2 x_3 \\ x_1 + x_3 \end{bmatrix}$$

is linear or not.

(b) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with

$$\text{Ker}(L) = \left\{ \begin{bmatrix} t + r \\ -t \\ 2t + r \end{bmatrix},\ t, r \in \mathbb{R} \right\}.$$

Is L onto? Find the rank and nullity of L.

(c) Let $L : V \to W$ be a linear transformation and let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in V. Prove that if $T = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ is linearly independent, then so is S. Explain.

(d) Let $L :\ P_2 \to \mathbb{R}^3$ be an one-to-one linear transformation. Is L onto? Justify your answer.

Question 5. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Let $S = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ be ordered basis for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

(a) Find the matrix A representing L with respect to S and T.

(b) If $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, compute $L(v)$ directly and then using A.