ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 205 - Basic Linear Algebra

FINAL EXAMINATION
24.05.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		20
2		25
3		20
4		20
5		25
Total		110

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Let V be the Euclidean space $\mathbb{R}^3$ with the standart inner product. Let

$$S = \{v_1 = (3, 0, 4),\ v_2 = (4, 3, 2),\ v_3 = (5, 2, -10)\}.$$

(a) Show that S is a basis for $\mathbb{R}^3$.

(b) Apply Gram-Schmidt process to transform the basis S into an orthogonal basis T.

(c) If $\alpha = (0, 1, 2)$ find $[\alpha]_T$

Solution:

(a) Write the vectors as column matrices. $A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 3 & 2 \\ 4 & 2 & -10 \end{bmatrix}$ $\det(A) = -130 \neq 0$.

Since $\det(A) \neq 0$, A is invertible, then S is a basis.

(b)

$$\begin{aligned}
\alpha_1 &= v_1 = (3, 0, 4) \\
\alpha_2 &= v_2 - \frac{< v_2, \alpha_1 >}{\|\alpha_1\|^2}\alpha_1 = \left(\frac{8}{5}, 3, \frac{-6}{5}\right) \\
\alpha_3 &= v_3 - \frac{< v_3, \alpha_1 >}{\|\alpha_1\|^2}\alpha_1 - \frac{< v_3, \alpha_2 >}{\|\alpha_2\|^2}\alpha_2 = \left(\frac{24}{5}, -4, \frac{-18}{5}\right)
\end{aligned}$$

(c) $[\alpha]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow$

$$c_1 = \frac{< \alpha, \alpha_1 >}{\|\alpha_1\|^2} = \frac{8}{25}, \quad c_2 = \frac{< \alpha, \alpha_2 >}{\|\alpha_2\|^2} = \frac{3}{65}, \quad c_3 = \frac{< \alpha, \alpha_3 >}{\|\alpha_3\|^2} = \frac{-14}{65}$$

Then $[\alpha]_T = \begin{bmatrix} 8/25 \\ 3/65 \\ -14/65 \end{bmatrix}$

Question 2. Given the linear transformation $L : \mathbb{R}^5 \to \mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 - 2x_5 \\ x_2 + 2x_3 - x_4 + x_5 \\ x_4 - x_5 \end{bmatrix}$$

(a) Find a basis for $\text{Ker}(L)$.

(b) Find a basis for $\text{Range}(L)$.

(c) Find $\text{nullity}(L)$ and $\text{rank}(L)$.

(d) Is L one-to-one and onto? Justify your answer.

(e) Is $v = (0, -5, 2, 2, 3)$ in $\text{Ker}(L)$?

Solution:

(a) $\alpha \in \text{Ker}(L) \Rightarrow L(\alpha) = 0$. Then

$$\begin{aligned} x_1 + 3x_3 - 2x_5 &= 0 \\ x_2 + 2x_3 - x_4 + x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t + 2r \\ -2t \\ t \\ r \\ r \end{bmatrix} = t\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \text{ Then } \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ is a basis for Ker}(L).$$

(b) $\beta \in \text{Range}(L)$ if

$$\begin{aligned} \beta = L(\alpha) &= \begin{bmatrix} x_1 + 3x_3 - 2x_5 \\ x_2 + 2x_3 - x_4 + x_5 \\ x_4 - x_5 \end{bmatrix} \\ &= x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + x_5\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

By (a) $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis for $\text{Range}(L)$.

(c) $\text{nullity}(L) = 2$ $\text{rank}(L) = 3$.

(d) L is not one-to-one since $\dim(\text{Ker}(L)) \neq 0$.
L is onto since $\dim(\text{Range}(L)) = \dim(\mathbb{R}^3) = 3$.

(e) Since $L(v) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \neq 0_{\mathbb{R}^3}$, $v \notin \text{Ker}(L)$

Question 3. Let $L : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation such that

$$L(p(x)) = 2p(x) + p'(x).$$

Let $\quad S = \{v_1 = 2 - 3x + x^2, v_2 = 1 - x^2, v_3 = -1 + x - 2x^2\}\quad$ and
$T = \{w_1 = 1 + 2x + 3x^2, w_2 = -x + x^2, w_3 = 3 + 5x + 11x^2\}\quad$ be ordered bases for $\mathbb{P}_2$.

(a) Find the matrix A representing L with respect to S and T.

(b) If $v = v_1 + 5v_3$, compute $L(v)$ in terms of w_1, w_2 and w_3.

Solution:

(a) $A = \left[\ [L(v_1)]_T \mid [L(v_2)]_T \mid [L(v_3)]_T\ \right]$

$L(v_1) = 1 - 4x + 2x^2, \quad L(v_2) = 2 - 2x - 2x^2, \quad L(v_3) = -1 - 2x - 4x^2$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 3 & 1 & 2 & -1 \\ 2 & -1 & 5 & -4 & -2 & -2 \\ 3 & 1 & 11 & 2 & -2 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 22 & 44 & 2 \\ 0 & 1 & 0 & 13 & 20 & 1 \\ 0 & 0 & 1 & -7 & -14 & -1 \end{array}\right] \Longrightarrow A = \begin{bmatrix} 22 & 44 & 2 \\ 13 & 20 & 1 \\ -7 & -14 & -1 \end{bmatrix}$$

(b) $[v]_S = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$. Then $[L(v)]_T = A[v]_S = \begin{bmatrix} 32 \\ 18 \\ -12 \end{bmatrix} \Longrightarrow L(v) = 32w_1 + 18w_2 - 12w_3$

Question 4.
(a) Determine whether $L : C[0,1] \to C[0,1]$ defined by $L(f) = \int_0^x f(t)dt \quad 0 \le x \le 1$ is linear or not.

Solution:

$$L(\alpha f + g) = \int_0^x (\alpha f(t) + g(t))dt = \alpha \int_0^x f(t)dt + \int_0^x g(t)dt = \alpha L(f) + L(g)$$

So L is a linear transformation.

(b) Determine whether $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 + x_2 \end{bmatrix}$ is linear or not.

Solution:

Let $\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\beta = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ then $L(\alpha) + L(\beta) = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ but $L(\alpha + \beta) = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix}$.
So L is not a linear transformation.

(c) Let $L : V \to W$ be a linear transformation. Prove that if $\dim V = \dim W$, then L is one-to-one if and only if L is onto.

Solution:

We know that $\text{nulltiy} L + \text{rank} L = \dim V$.

If L is one-to-one then $\text{nullity}(L) = 0$. Then $\text{rank} L = \dim V = \dim W$, so L is onto.

If L is onto, then $\text{rank} L = \dim W = \dim V$. Then $\text{nulltiy} L = 0$ which means that L is one-to-one.

(d) Let $L : \mathbb{P}_3 \to \mathbb{P}_2$ be a linear transformation with

$$\text{Range}(L) = \{ax^2 + ax + a \mid a \in \mathbb{R}\}.$$

Is L one-to-one? Find the rank and nullity of L.

Solution:

$\beta \in \text{Range}(L) \Rightarrow \beta = ax^2 + ax + a = a(x^2 + x + 1)$.

So $x^2 + x + 1$ is a basis for $\text{Range}(L)$ and $\text{rank}(L) = 1$.

Then $\text{nulltiy} L = \dim \mathbb{P}_3 - \text{rank} L = 4 - 1 = 3$. Since $\text{nulltiy} L \neq 0$, L is not one-to-one.

Question 5.

(a) Let $L : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation such that

$$\begin{aligned}
L(p_1) &= L\left(1 + x + 2x^2\right) = 1 + 2x + 3x^2 \\
L(p_2) &= L\left(-1 - x^2\right) = x + x^2 \\
L(p_3) &= L\left(3 + 2x + 5x^2\right) = 1 + x \\
L(p_4) &= L\left(-1 - x - x^2\right) = -2 + 3x + 4x^2
\end{aligned}$$

Compute $L\left(a + bx + cx^2\right)$.

Solution:

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & a \\ 1 & 0 & 2 & -1 & b \\ 2 & -1 & 5 & -1 & c \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & c-a \\ 0 & 1 & -1 & 0 & b-a \\ 0 & 0 & 0 & 1 & c-a-b \end{array}\right]$$

Then p_1, p_2 and p_4 is a basis for $\mathbb{P}_2$ and

$$\begin{aligned}
L(a + bx + cx^2) &= (c-a)L(p_1) + (b-a)L(p_2) + (c-a-b)L(p_4) \\
&= a + 2b - c + (-6a - 2b + 5c)x + (-8a - 3b + 7c)x^2
\end{aligned}$$

(b) Solve the following linear system by Cramer's rule.

$$\begin{aligned}
x_1 + 2x_2 - x_3 &= 1 \\
2x_1 + 3x_2 - x_3 &= 0 \\
-3x_1 - 4x_2 + 2x_3 &= 3
\end{aligned}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ -3 & -4 & 2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 3 & -4 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ -3 & 3 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ -3 & -4 & 3 \end{bmatrix}$$

Then $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{5}{-1} = -5$, $x_2 = \dfrac{|A_2|}{|A|} = \dfrac{-4}{-1} = 4$, $x_3 = \dfrac{|A_3|}{|A|} = \dfrac{-2}{-1} = 2$