ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MCS 205 - Basic Linear Algebra

FIRST MIDTERM EXAMINATION
13.03.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		25
5		20
Total		105

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 - 2x_5 &= 0 \\ 2x_1 + 3x_2 + 4x_3 + 4x_4 - x_5 &= 3 \\ 3x_1 + 2x_2 + x_3 + 10x_4 + 4x_5 &= 10 \\ -2x_2 - 4x_3 + 4x_4 + 6x_5 &= 6 \end{aligned}$$

(a) Form the augmented matrix of the given system.

(b) Find the (reduced) row echelon form of the augmented matrix, which is found in part (a).

(c) Solve the system if it is consistent.

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -2 & 0 \\ 2 & 3 & 4 & 4 & -1 & 3 \\ 3 & 2 & 1 & 10 & 4 & 10 \\ 0 & -2 & -4 & 4 & 6 & 6 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -6 & -4 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4 + t + 6r \\ 1 - 2t - r \\ t \\ 2 - 2r \\ r \end{bmatrix} \qquad \text{where } t, r \in \mathbb{R}.$$

2) Find $a, b \in \mathbb{R}$ for which the following system has,

(a) no solution, **(b)** infinitely many solutions, **(c)** a unique solution

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 4x_4 &= 0 \\ x_2 + 2x_3 + 5x_4 &= -2 \\ x_2 + (a-1)x_3 + 6x_4 &= b+1 \\ (a-3)x_3 + (a-1)x_4 &= b+4. \end{aligned}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & 1 & a-1 & 6 & b+1 \\ 0 & 0 & a-3 & a-1 & b+4 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & 0 & a-3 & 1 & b+3 \\ 0 & 0 & 0 & a-2 & 1 \end{array}\right]$$

- $a = 2$ no solution.
- $a = 3$ and $b \neq -2$ no solution.
- $a = 3$ and $b = -2$ infinitely many solution.
- $a \neq 3$ and $a \neq 2$ unique solution.

3) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

(a) Find A^{-1} by using elementary row operations.

(b) Find $\text{adj}(A)$.

(c) Solve the system $Ax = b$ if $b = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

Solution:

(a) $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

(b) $\text{adj}(A) = \det(A)A^{-1}$ and $\det(A) = -1 \Longrightarrow \text{adj}(A) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

(c) $x = A^{-1}b = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$

4) Let $A = \begin{bmatrix} -1 & -4 & -8 & -2 \\ 2 & -1 & 7 & 9 \\ -1 & 5 & 3 & -1 \\ 1 & -2 & 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & -2 & 0 & 1 & 4 \\ 1 & 2 & 7 & 3 & 1 \\ 0 & 4 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -4 & -8 \\ 2 & -1 & 0 \\ -1 & 1 & 3 \end{bmatrix}$,

$$D = \begin{bmatrix} 1 & -4 & -8 \\ 2 & -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 4 & -8 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}.$$

Compute the following determinants if it is possible:
(a) $|A|$
(b) $|B|$
(c) $|C|$
(d) $|D|$
(e) $|DE|$
(f) $|C^3|$
(g) $|C^T(DE)^{-1}|$
(h) $\left|\left(\frac{3}{2}C\right)^{-1}\right|$

Solution:

(a) $$|A| = \begin{vmatrix} -1 & -4 & -8 & -2 \\ 2 & -1 & 7 & 9 \\ -1 & 5 & 3 & -1 \\ 1 & -2 & 0 & -4 \end{vmatrix} = \begin{bmatrix} -1 & -4 & -8 & -2 \\ 0 & -9 & -9 & 5 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & -10/3 \end{bmatrix} = (-1)(-9)2\frac{-10}{3} = -60$$

(b) $|B| = 168$

(c) $|C| = 19$

(d) It is not possible to compute $|D|$ since D is not a square matrix.

(e) $$|DE| = \begin{vmatrix} -8 & 32 \\ 7 & -16 \end{vmatrix} = 352$$

(f) $|C^3| = |C|^3 = 19^3$

(g) $$|C^T(DE)^{-1}| = \frac{|C|}{|DE|} = \frac{19}{352}$$

(h) $$\left|\left(\frac{3}{2}C\right)^{-1}\right| = \left(\frac{2}{3}\right)^3 \frac{1}{19}$$

5)

(a) Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & k & 2 \\ 0 & 1 & (k-1) \end{bmatrix}$

Find all values of k if the system $AX = 0$ has only trivial solution.

Solution:

If the system has only trivial solution then $\det(A) \neq 0$.

Since $\det(A) = k^2 - k - 2$, $k \in \mathbb{R}/\{-1, 2\}$

(b) Find a matrix X such that $\begin{bmatrix} 1 & 2 \\ 5 & 2 \\ 0 & 1 \end{bmatrix} = X \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Solution:

If $A = XB$ then $X = AB^{-1} = \begin{bmatrix} 1 & 2 \\ 5 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -13 \\ 0 & 1 \end{bmatrix}$

(c) Let $B = \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}$. Find an invertible 2×2 matrix A such that $A^3 = A^2B - 3A^2$.

Solution:

$$A^3 = A^2(B - 3I) \Longrightarrow A = B - 3I = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

(d) Suppose that an $n \times n$ matrix satisfies the equation

$$A^4 + 2A^3 - 3A^2 - A + I = 0.$$

Show that A is invertible and give a formula for A^{-1} in terms of A.

Solution:

$$I = -A^4 - 2A^3 + 3A^2 + A = A\underbrace{(-A^3 - 2A^2 + 3A + I)}_{A^{-1}}$$

Then A is invertible and $A^{-1} = -A^3 - 2A^2 + 3A + I$.