ÇANKAYA UNIVERSITY

Department of Mathematics

MCS 205 - Basic Linear Algebra

SECOND MIDTERM EXAMINATION
17.04.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		20
2		20
3		25
4		20
5		20
Total		105

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 5 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. **(a)** Let W be the subspace of $\mathbb{R}^3$ spanned by the following vectors

$$v_1 = (1,2,3,0), \qquad v_2 = (2,3,2,-2), \qquad v_3 = (3,4,1,-4), \qquad v_4 = (1,4,10,4)$$

Find a basis S for W.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 4 \\ 3 & 2 & 1 & 10 \\ 0 & -2 & -4 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\{v_1, v_2, v_4\}$ is a basis for W.

(b) If $S = \{v_1, v_2, \ldots, v_n\} \subset V$, define span$(S)$.

span(S) is the set of all linear combinations of $\{v_1, v_2, \ldots, v_n\}$.

(c) If $S = \{v_1, v_2, \ldots, v_n\} \subset V$, define what it means for S to be linearly independent.

S is linearly independent if the equation $c_1v_1 + \cdots + c_nv_n = 0$ has only trivial solution $c_1 = \cdots = c_n = 0$.

(d) If $S = \{v_1, v_2, \ldots, v_n\} \subset V$, define what it means for S to be a basis of V.

S is basis if S is linearly independent and span$(S) = V$.

(e) Consider the basis $S = \{t^2 - t + 1, t + 1, -3t^2 - 3\}$ for the space of polynomials of degree at most 2. If $[v]_S = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, find v.

$$v = t^2 - t + 1 + (-2)(t+1) + 4(-3t^2 - 3) = -11t^2 - 3t - 13.$$

(f) Consider the basis $S = \{(1,3),(2,4)\}$ of $\mathbb{R}^2$. If $v = (3,5)$ find $[v]_S$.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \Longrightarrow [v]_S = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 2. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -2 & 4 & 8 & 8 \\ -2 & 4 & 3 & -8 & -6 & -9 & -9 \\ 0 & 0 & 2 & -8 & 1 & 4 & 6 \\ -1 & -2 & 3 & -10 & -1 & 1 & -5 \end{bmatrix}$$

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the null space of A.

(d) Find the rank and the nullity of A.

(e) Find a basis for $\mathbb{R}^4$ containing the basis elements for the the column space.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 & -2 & 4 & 8 & 8 \\ -2 & 4 & 3 & -8 & -6 & -9 & -9 \\ 0 & 0 & 2 & -8 & 1 & 4 & 6 \\ -1 & -2 & 3 & -10 & -1 & 1 & -5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 24 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & -4 \end{bmatrix}$$

(a) All rows of A form a basis for the row space of A.

(b) $c_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, $c_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$, $c_5 = \begin{bmatrix} 4 \\ -6 \\ 1 \\ -1 \end{bmatrix}$ form a basis for the column space of A.

(c) $\begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -24 \\ 0 \\ -5 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

(d) rank(A)=4 and nullity(A)=3.

(e) Since c_1, c_2, c_3, c_5 are linearly independent (since basis) and $\dim(\mathbb{R}^4) = 4$, c_1, c_2, c_3, c_5 are also basis for $\mathbb{R}^4$.

Question 3.
(a) Consider the basis $S = \{v_1 = (1, -2, 4),\ v_2 = (0, 1, 2),\ v_3 = (-1, 1, 5)\}$ and $T = \{w_1 = (-1, 3, 2),\ w_2 = (1, 2, 1),\ w_3 = (0, 1, -2)\}$.

i) Find a transition matrix $P_{S \leftarrow T}$ from T to S.

ii) If $\alpha = 2w_1 - w_2 + 3w_3$, write α as a linear combination of v_i's.

Solution:

$$P_{S \leftarrow T} = \left[\ [w_1]_S \mid [w_2]_S \mid [w_3]_S\ \right]$$

$$A = \left[\begin{array}{ccc|c|c|c} 1 & 0 & -1 & -1 & 1 & 0 \\ -2 & 1 & 1 & 3 & 2 & 1 \\ 4 & 2 & 5 & 2 & 1 & -2 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -7/11 & 0 & -4/11 \\ 0 & 1 & 0 & 15/11 & 3 & 7/11 \\ 0 & 0 & 1 & 4/11 & -1 & -4/11 \end{array}\right]$$

Then $P_{S \leftarrow T} = \begin{bmatrix} -7/11 & 0 & -4/11 \\ 15/11 & 3 & 7/11 \\ 4/11 & -1 & -4/11 \end{bmatrix}$

$$[\alpha]_T = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad [\alpha]_S = P_{S \leftarrow T}[\alpha]_T = \begin{bmatrix} -26/11 \\ 18/11 \\ 7/11 \end{bmatrix}$$

$$\alpha = \frac{-26}{11}v_1 + \frac{18}{11}v_2 + \frac{7}{11}v_3$$

(b) If $C = \{v_1, v_2, v_3\}$, $D = \{w_1,\ w_2,\ w_3\}$ is a basis for the vector space V and

$$P_{C \leftarrow D} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

is the transition matrix from D to C, write down the vectors v_1, v_2 and v_3 as a linear combination of the vectors $w_1,\ w_2,\ w_3$.

Solution:

$$P_{D \leftarrow C} = P_{C \leftarrow D}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -3 & -2 \\ 2 & -1 & -1 \end{bmatrix} = \left[\ [v_1]_D \mid [v_2]_D \mid [v_3]_D\ \right]$$

$v_1 = w_1 + 6w_2 + 2w_3,\ v_2 = -3w_2 - w_3,\ v_3 = -2w_2 - w_3$

Question 4.

(a) For which values of k are the vectors $\{(1,2,3),\ (0,k,3),\ (k-1,1,3)\}$ linearly dependent.

Solution:

This set is linearly dependent if the matrix $A = \begin{bmatrix} 1 & 0 & k-1 \\ 2 & k & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is not invertible, that is, $\det(A) = 0$.

$\det(A) = -3k^2 + 12k - 9 = 0 \Longrightarrow k = 3$ and $k = 1$.

(b) Let A and B are subspaces of V.

i) Prove that $A + B := \{a + b \in V \mid a \in A,\ b \in B\}$ is a subspace of V.

ii) Prove that $A \cap B$ is a subspace of V.

iii) (BONUS) Prove that $A \cup B$ is not necessarily a subspace of V and if it subspace of V then $A \subseteq B$ or $B \subseteq A$.

Solution:

i) Let $a_1 + b_1$ and $a_2 + b_2 \in A + B$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then

$$k(a_1 + b_1) + a_2 + b_2 = ka_1 + a_2 + kb_1 + b_2 \in A + B$$

since $ka_1 + a_2 \in A$ (A is a subspace of V) and $kb_1 + b_2 \in B$ (B is a subspace of V). Then $A + B$ is a subspace of V.

ii) Let $u, v \in A \cap B$. Then $u, v \in A$ and $u, v \in B$.
$u, v \in A$ then $ku + v \in A$ since A is a subspace of V.
$u, v \in B$ then $ku + v \in B$ since B is a subspace of V.
So $ku + v \in A \cap B$ and $A \cap B$ is a subspace of V.

iii) Let $V = \mathbb{R}^2$. $A = \{(a,0) \mid a \in \mathbb{R}\}$ and $B = \{(0,b) \mid b \in \mathbb{R}\}$ are subspaces of V. Since $(1,0) + (0,1) = (1,1) \notin A \cup B$, $A \cup B$ is not a subspace of V.

If $A \cup B$ subspace of V then $A \subseteq B$ or $B \subseteq A$. Assume it is not true. There exists $u \in A - B$ and $v \in B - A$.. Then $u + v \in A \cup B$.
If $u + v \in A$ then $u + v - u = v \in A$ contradicts our assumption.
If $u + v \in B$ then $u + v - v = u \in B$ contradicts our assumption.
So $A \subseteq B$ or $B \subseteq A$.

(c) Let $\{v_1, v_2, v_3\}$ is a basis for a vector space V. Determine whether the vectors $v_1 + v_2,\ v_2 + v_3,\ v_1 - v_3$ form a basis for V.

Solution:

Linear independence: $c_1(v_1 + v_2) + c_2(v_2 + v_3) + c_3(v_1 - v_3) = 0$ has only trivial solution $c_1 = c_2 = c_3 = 0$. Then $v_1(c_1 + c_3) + v_2(c_1 + c_2) + v_3(c_2 - c_3) = 0$ and since $\{v_1, v_2, v_3\}$ is a basis

$$c_1 + c_3 = 0, \quad c_1 + c_2 = 0 \quad c_2 - c_3 = 0.$$

This system has only trivial solution if the coefficient matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ is invertible. Since $\det(A) = 0$, A is not invertible and these vectors not form a basis.

Question 5.

(a) Show that the set of vectors of the form $\begin{bmatrix} a \\ a+b \\ 2a-b \end{bmatrix}$ forms a subspace of $\mathbb{R}^3$. Find a basis for this space. What is it's dimension?

Solution:

Let $W =$ the set of vectors of the form $\begin{bmatrix} a \\ a+b \\ 2a-b \end{bmatrix}$.

Let $\alpha = \begin{bmatrix} a \\ a+b \\ 2a-b \end{bmatrix}$, $\beta = \begin{bmatrix} c \\ c+d \\ 2c-d \end{bmatrix} \in W$ and $k \in \mathbb{R}$. Then

$$\alpha+\beta = \begin{bmatrix} a+c \\ a+b+c+d \\ 2a-b+2c-d \end{bmatrix} = \begin{bmatrix} a+c \\ a+c+b+d \\ 2(a+c)-(b+d) \end{bmatrix} \in W \text{ and } k\alpha = \begin{bmatrix} ka \\ ka+kb \\ 2ka-kb \end{bmatrix} \in W$$

So W is a subspace of $\mathbb{R}^3$. Also since

$$\begin{bmatrix} a \\ a+b \\ 2a-b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a basis for W and $\dim(W) = 2$.

(b) Let $W = \left\{ f \in C[0,1] \;\middle|\; \int_0^1 |f(x)|dx < \infty \right\}$. Is W a subspace of $C[0,1]$?
($C[0,1]$ is the space of continuous functions on $[0,1]$)

Solution:

Let $f, g \in W$ and $k \in \mathbb{R}$. Then

$$\int_0^1 |f(x)|dx < \infty, \quad \int_0^1 |g(x)|dx < \infty.$$

$$\int_0^1 |kf(x)+g(x)|dx \le \int_0^1 (k|f(x)|+|g(x)|)dx = k\int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx < \infty$$

So $kf+g \in W$ and W is a subspace of $C[0,1]$.