Exercise Set I

1. Solve the linear system below:

$$\begin{array}{rcrcrcrcr} x & + & y & + & 2z & - & w & = & 0 \\ 2x & + & 2y & + & 2z & - & w & = & -1 \\ & & y & & & - & 2w & = & 0 \\ 4x & - & 2y & & & - & 2w & = & 1. \end{array}$$

2. Solve the system:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 0 \\ 2x_1 & - & 3x_2 & + & x_3 & - & 8x_4 & = & 1. \end{array}$$

3. Solve the system:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & - & 3x_4 & = & 2 \\ & & 2x_2 & + & x_3 & + & x_4 & = & 0. \end{array}$$

4. Let

$$A = \begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$$

be the augmented matrix of some linear system of equations. For which values of "a" and "b" does the system have

a) a unique solution?
b) a one-parameter solution?
c) a two-parameter solution?
d) no solution?

5. Solve the systems of linear equations by Gaussian and by Gauss-Jordan eliminations

a)
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & x_3 & = & 8 \\ 6x_1 & - & 5x_2 & - & 3x_3 & = & 55 \\ 7x_1 & - & 6x_2 & - & 4x_3 & = & 63 \end{array}$$

b)
$$\begin{array}{rcrcrcrcr} 3x_1 & - & 3x_2 & + & 12x_3 & + & 3x_4 & = & 6 \\ 3x_1 & + & 2x_2 & + & 12x_3 & + & 8x_4 & = & 46 \\ x_1 & & & + & 4x_3 & + & 2x_4 & = & 10 \\ 9x_1 & - & 4x_2 & + & 36x_3 & + & 14x_4 & = & 58 \end{array}$$

c)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -x_1 & + & 2x_2 & + & x_3 & = & -2 \\ x_1 & + & x_2 & + & 2x_3 & = & -1 \end{array}$$

d)
$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 1 \\ 5x_1 & + & 3x_2 & + & 2x_3 & = & 3 \\ x_1 & + & x_2 & & & = & 1 \end{array}$$

6. For which values of a does the system

$$\begin{array}{rcrcr} 3x & + & ay & = & 0 \\ (a-2)x & + & 5y & = & 0 \end{array}$$

have

a) a unique solution?
b) infinitely many solutions?

7) Solve the following systems, where a, b, and c are constants.

a)
$$\begin{array}{rcrl} 2x & + & y & = a \\ 3x & + & 6y & = b \end{array}$$

b)
$$\begin{array}{rcrcrl} x_1 & + & x_2 & + & x_3 & = a \\ 2x_1 & & & + & 2x_3 & = b \\ & & 3x_2 & + & 3x_3 & = c \end{array}$$

8) For which values of a will the following system have no solutions? Exactly one solutions? Infinitely many solutions?

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2. \end{array}$$

9) Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$ and $0 \le \gamma \le \pi$.

$$\begin{array}{rcrcrcr} 2\sin\alpha & - & \cos\beta & + & 3\tan\gamma & = & 3 \\ 4\sin\alpha & + & 2\cos\beta & - & 2\tan\gamma & = & 2 \\ 6\sin\alpha & - & 3\cos\beta & + & \tan\gamma & = & 9. \end{array}$$

Hint: First substitute $x_1 = \sin\alpha$, $x_2 = \cos\beta$ and $x_3 = \tan\gamma$ and solve the resulting linear system in the unknowns x_1, x_2 and x_3.

10) For which value(s) of λ does the system of equations

$$\begin{array}{rcrcl} (\lambda-3)x & + & y & = & 0 \\ x & + & (\lambda-3)y & = & 0 \end{array}$$

have nontrivial solutions?

11) Solve the following system for x, y and z.

$$\begin{array}{rcrcrcl} \frac{1}{x} & + & \frac{2}{y} & - & \frac{4}{z} & = & 1 \\ \\ \frac{2}{x} & + & \frac{3}{y} & + & \frac{8}{z} & = & 0 \\ \\ \frac{1}{x} & + & \frac{9}{y} & - & \frac{10}{z} & = & 5 \end{array}$$

Hint: Substitute $x_1 = \frac{1}{x}$, $x_2 = \frac{1}{y}$ and $x_3 = \frac{1}{z}$ and solve the linear system in the unknowns x_1, x_2 and x_3.

12) Show that if $ad - bc \neq 0$, then the reduced row-echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

13) For the following system of nonhomogeneous linear equations

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & -2 \\ & & x_2 & + & x_3 & - & x_4 & - & x_5 & = & -1 \\ -x_1 & + & x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & 2 \end{array}$$

a) Find the reduced row echelon form of the augmented matrix of this system;
b) Find its solution in the parametric form.

14) For which values of parameter λ the following homogeneous system of linear equations has a nontrivial solution?

$$\begin{aligned} \lambda x &- y &&= 0 \\ x &- 2y &+ \lambda z &= 0 \\ x &- y &- z &= 0. \end{aligned}$$

15)

a) Determine for which values of parameter a the following system of linear equations is consistent?

$$\begin{aligned} 2x_1 &+ x_2 &+ x_3 &+ 2x_4 &= 10 \\ x_1 &- x_2 &+ x_3 &- 3x_4 &= -20 \\ 3x_1 &+ 2x_2 &- x_3 &+ x_4 &= 16 \\ 5x_1 &+ 2x_2 &- 2x_3 &- 3x_4 &= a. \end{aligned}$$

b) For those values of the parameter a for which the system is consistent, find all solutions of the system.

16) For the following nonhomogeneous system of linear equations

$$\begin{aligned} x_1 &- x_2 &+ x_3 &- 2x_4 &= -1 \\ 2x_1 &- 2x_2 &+ 3x_3 &+ 3x_4 &= 2 \\ 4x_1 &- 4x_2 &+ 5x_3 &+ 7x_4 &= 0 \end{aligned}$$

a) Find the reduced row-echelon form of the augmented matrix of the system.
b) Find the solution of this system in the parametric form.

17) The following matrices are obtained as a result of row reduction of the augmented matrices of two systems of linear equations:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & -1 & 9 & 4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 3 & 4 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Complete the row reduction to obtain the reduced row echelon matrices;
b) Determine if the system is consistent;
c) For each consistent system find the solution in the parametric form.

18) Find the solution to the system of linear equations

$$\begin{aligned} x_1 &- 7x_4 &+ 8x_5 &= 0 \\ x_2 &+ 2x_3 &+ 2x_4 &= 0 \\ x_3 &+ x_4 &- x_5 &= 0 \end{aligned}$$

in the parametric form.