Exercise Set II

1. Consider the matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},\ D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix},\ E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Compute the following (where possible).

a) $D+E$ **b)** $D-E$ **c)** $5A$ **d)** $-7C$
e) $2B-C$ **f)** $4E-2D$ **g)** $-3(D+2E)$ **h)** $A-A$
i) $\mathrm{tr}(D)$ **j)** $\mathrm{tr}(D-3E)$ **k)** $4\mathrm{tr}(7B)$ **l)** $\mathrm{tr}(A)$

2. Using the matrices in Exercise 1, compute the following (where possible).

a) $2A^T+C$ **b)** D^T-E^T **c)** $(D-E)^T$ **d)** B^T+5C^T
e) $\frac{1}{2}C^T-\frac{1}{4}A$ **f)** $B-B^T$ **g)** $2E^T-3D^T$ **h)** $(2E^T-3D^T)^T$

3. Using the matrices in Exercise 1, compute the following (where possible).

a) AB **b)** BA **c)** $(3E)D$ **d)** $(AB)C$
e) $A(BC)$ **f)** CC^T **g)** $(DA)^T$ **h)** $(C^TB)A^T$
i) $\mathrm{tr}(DD^T)$ **j)** $\mathrm{tr}(4E^T-D)$ **k)** $\mathrm{tr}(C^TA^T+2E^T)$

4. Using the matrices in Exercise 1, compute the following (where possible).

a) $(2D^T-E)A$ **b)** $(4B)C+2B$ **c)** $(-AC)^T+5D^T$
d) $(BA^T-2C)^T$ **e)** $B^T(CC^T-A^TA)$ **f)** $D^TE^T-(ED)^T$

5. If A and B are partitioned into sub-matrices, for example,

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

then AB can be expressed as

$$AB = \left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

provided the sizes of the sub-matrices of A and B are such that the indicated operations can be performed. This method of multiplying partitioned matrices is called ***block multiplication***. In each part compute the product by block multiplication. Check your results by multiplying directly.

a) $$A = \left[\begin{array}{rr|rr} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 2 & 1 & 4 \\ -3 & 5 & 2 \\ \hline 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right].$$

b) $$A = \left[\begin{array}{rrr|r} -1 & 2 & 1 & 5 \\ \hline 0 & -3 & 4 & 2 \\ 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 2 & 1 & 4 \\ -3 & 5 & 2 \\ 7 & -1 & 5 \\ \hline 0 & 3 & -3 \end{array}\right].$$

6. Find the 4×4 matrix $A=[a_{ij}]$ whose entries satisfy the stated condition.

a) $a_{ij}=i+j$ **b)** $a_{ij}=i^{j-1}$ **c)** $a_{ij}=\begin{cases} 1 \text{ if } |i-j|>1 \\ -1 \text{ if } |i-j|\le 1 \end{cases}$

7. Prove: If A and B are $n\times n$ matrices, then $\mathrm{tr}(A+B)=\mathrm{tr}(A)+\mathrm{tr}(B)$.

8. How many 3×3 matrices A can you find such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$$

for all choices of x, y, and z?

9. A matrix B is said to be a ***square root*** of a matrix A if $BB=A$.

a) Find two square roots of $A=\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

b) How many different square roots can you find of $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

10. Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$. Show:

a) $(AB)^{-1} = B^{-1}A^{-1}$ **b)** $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

11. In each part use the given information to find A.

a) $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ **b)** $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

c) $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ **c)** $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. In each part find $p(A)$.

a) $p(x) = x - 2$ **b)** $p(x) = 2x^2 - x + 1$ **c)** $p(x) = x^3 - 2x + 4$

13. Find the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

14. Find the inverse of $\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$.

15. Let A and B be square matrices such that $AB = 0$. Show that if A is invertible, then $B = 0$.

16. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

17. Let A, B, and 0 be 2×2 matrices. Assuming that A is invertible, find a matrix C so that

$$\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline C & A^{-1} \end{array}\right]$$

is the inverse of the partitioned matrix

$$\left[\begin{array}{c|c} A & 0 \\ \hline B & A \end{array}\right].$$

18. Show that if A is invertible and $AB = AC$, then $B = C$.

19. Which of the following are elementary matrices?

a) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **f)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ **g)** $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices, E_1, E_2, E_3 and E_4 such that

a) $E_1A = B$ **b)** $E_2B = A$ **c)** $E_3A = C$ **d)** $E_4C = A$

21. Find the inverses of the matrices, if any:

a) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$, **b)** $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **c)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

d) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$ **e)** $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

22. Solve the following general system by inverting the coefficient matrix.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= b_1 \\ x_1 - x_2 + x_3 &= b_2 \\ x_1 + x_2 &= b_3 \end{aligned}$$

Use the resulting formulas to find the solution if
a) $b_1 = -1,\ b_2 = 3,\ b_3 = 4$ **b)** $b_1 = 5,\ b_2 = 0,\ b_3 = 0$ **c)** $b_1 = -1,\ b_2 = -1,\ b_3 = 3$

23. Find conditions that b's must satisfy for the systems below to be consistent.

a) $\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$ **b)** $\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$

c) $\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$ **d)** $\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_1 &= b_4 \end{aligned}$

24. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a) Show that the equation $Ax = x$ can be rewritten as $(A - I)x = 0$ and use this result to solve $Ax = x$ for x.
b) Solve $Ax = 4x$.

25. Solve the following matrix equation for X.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

Hint:
Method 1) Denote

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \end{bmatrix}$$

Then, reduce the matrix equation to a system of linear equations in 15 unknown and then solve by Gauss-Jordan (or Gaussian) elimination.
Method 2) Find A^{-1} if any, and then multiply both sides of the equality by A^{-1} (from left).

26. Find all values of a, b and c for which A is symmetric.

$$A = \begin{bmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 7 \end{bmatrix}$$

27. Find all values of a and b for which A and B are both not invertible.

$$A = \begin{bmatrix} a+b-1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 2a-3b-7 \end{bmatrix}$$

28. Fact: The product of two symmetric matrices is symmetric if and only if the matrices commute. Consider

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Note that A, B, C and D are symmetric matrices. Show that AB is not symmetric while CD is symmetric. Are they true that $AB \neq BA$ and $CD = DC$?

29. Show that A and B commute if $a - d = 7b$. (*Hint: Use the fact in Exercise 28*).

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

30. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Determine whether A is symmetric.

a) $a_{ij} = i^2 + j^2$ **b)** $a_{ij} = i^2 - j^2$

c) $a_{ij} = 2i + 2j$ **d)** $a_{ij} = 2i^2 + 2j^3$

31) Find positive integers that satisfy

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 9 \\ x & + & 5y & + & 10z & = & 44. \end{array}$$

32. Solve for x, y and z.

$$\begin{array}{rcrcrl} xy & - & 2\sqrt{y} & + & 3zy & = 8 \\ 2xy & - & 3\sqrt{y} & + & 2zy & = 7 \\ -xy & + & \sqrt{y} & + & 2zy & = 4 \end{array}$$

33. Find a matrix K such that $AKB = C$ given that

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

34. How should the coefficients a, b and c be chosen so that the system

$$\begin{array}{rcrcrcr} ax & + & by & - & 3z & = & -3 \\ -2x & - & by & + & cz & = & -1 \\ ax & + & 3y & - & cz & = & -3 \end{array}$$

has the solution $x = 1$, $y = -1$ and $z = 2$?

35. Find values of a, b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the point $(-1, 0)$ and has a horizontal tangent at $(2, -9)$.

36. Find the values of A, B and C that will make the equation

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx + C}{x^2+1}.$$

an identity.

37. Consider the matrices, $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find

a) $2A - B^T$.

b) A^{-1}.

c) Use A^{-1} found in part **b)** to solve the system $Ax = b$.

38. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$, $C = AB$. Let $p(x) = 3 + 2x^2 - x^3$. Find $P(C)$.

Note: If A is a square matrix, and if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is any polynomial, then we define $p(A) = a_0 I + a_1 A + \cdots + a_n A^n$ where I is the identity matrix of suitable size.