Exercise Set III

1. Find all value(s) of λ for which $|A| = 0$.

a) $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$ **b)** $\begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$

2. Solve for x.

$$\begin{vmatrix} x & -1 \\ 3 & 1 - x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x - 5 \end{vmatrix}$$

3. Show that the value of the determinant

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

does not depend on θ.

4. Prove that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0.$$

5. Evaluate the following determinants by inspection.

a) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 6 & 1 \end{vmatrix}$ **b)** $\begin{vmatrix} -2 & 1 & 3 \\ 1 & -7 & 4 \\ -2 & 1 & 3 \end{vmatrix}$ **c)** $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 5 & -8 & 1 \end{vmatrix}$ **d)** $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

6. In each part, evaluate the determinant of the given matrix by reducing the matrix to row-echelon form.

a) $\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$ **d)** $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

e) $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$ **f)** $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1/2 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{bmatrix}$ **h)** $\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

7. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ **b)** $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$ **c)** $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$ **d)** $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

8. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

9. By inspection, solve the equation

$$\begin{vmatrix} x & 5 & 7 \\ 0 & x+1 & 6 \\ 0 & 0 & 2x-1 \end{vmatrix} = 0.$$

Explain your reasoning.

10. a) By inspection, find two solutions of the equation

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & -3 & 9 \end{vmatrix} = 0.$$

b) Is it possible that there are other solutions? Justify your answer.

11. Determine which of the following matrices are invertible.

a) $A = \begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$

12. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Assuming that $|A| = -7$, find

a) $|3A|$ **b)** $|A^{-1}|$ **c)** $|2A^{-1}|$ **d)** $|(2A)^{-1}|$ **e)** $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

13. Without directly evaluating, show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

14. For which value(s) of k does A fail to be invertible?

a) $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$

15. Show that

$$\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$$

is not invertible for any values of α, β and γ.

16. Prove that a square matrix A is invertible if and only if $A^T A$ is invertible.

17. Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

a) Find all the minors of A.
b) Find all the cofactors.

18. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

Find
a) M_{13} and A_{13} **b)** M_{23} and A_{23} **c)** M_{22} and A_{22} **d)** M_{21} and A_{21}

19. For the matrix in Exercise 17, find
a) $\mathrm{adj}(A)$,
b) A^{-1}.

20. Evaluate $|A|$ by a cofactor expansion along a row or column of your choice.

a) $\begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$ **b)** $\begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

d) $\begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$ **e)** $\begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$ **f)** $\begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

21. Find A^{-1} using $\mathrm{adj}(A)$.

a) $\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$ **d)** $\begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$

22. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}.$$

a) Evaluate A^{-1} using Gauss-Jordan elimination.
b) Evaluate A^{-1} using $\mathrm{adj}(A)$.
c) Which method involves less computation?

23. Solve by Cramer's rule, where it applies.

a) $\begin{array}{rcrcrcr} 4x & + & 5y & & & = & 2 \\ 11x & + & y & + & 2z & = & 3 \\ x & + & 5y & + & 2z & = & 1 \end{array}$

b) $\begin{array}{rcrcrcrcr} -x_1 & - & 4x_2 & + & 2x_3 & + & x_4 & = & -32 \\ 2x_1 & - & x_2 & + & 7x_3 & + & 9x_4 & = & 14 \\ -x_1 & + & x_2 & + & 3x_3 & + & x_4 & = & 11 \\ x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & -4 \end{array}$

c) $\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 4 \\ -x_1 & + & 7x_2 & - & 2x_3 & = & 1 \\ 2x_1 & + & 6x_2 & - & x_3 & = & 5 \end{array}$

24. Use Cramer's rule to solve for y without solving for x, z and w.

$$\begin{array}{rcrcrcrcr} 4x & + & y & + & z & + & w & = & 6 \\ 3x & + & 7y & - & z & + & w & = & 1 \\ 7x & + & 3y & - & 5z & + & 8w & = & -3 \\ x & + & y & + & z & + & 2w & = & 3 \end{array}$$

25. a) If $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$ is an "upper triangular" block matrix, where A_{11} and A_{22} are square matrices, then $|A| = |A_{11}||A_{22}|$. Use this result to evaluate $|A|$ for

$$\left[\begin{array}{cc|ccc} 2 & -1 & 2 & 5 & 6 \\ 4 & 3 & -1 & 3 & 4 \\ \hline 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & -2 & 6 & 2 \\ 0 & 0 & 3 & 5 & 2 \end{array}\right]$$

b) Verify your answer in part **a)** by using a cofactor expansion to evaluate $|A|$.

26. Let

$$A = \begin{bmatrix} 2 & -2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 6 \end{bmatrix}.$$

a) Find $|A|$.
b) Is A invertible? Give reasons for your answer.
c) Let $B = \text{adj}(A)$. Calculate b_{12}.

27. For which value(s) of λ the following homogeneous system of linear equations has a nontrivial solution?

$$\begin{array}{rcrcrcl} \lambda x & - & y & & & = & 0 \\ x & - & 2y & + & \lambda z & = & 0 \\ x & - & y & - & z & = & 0 \end{array}$$

28. For which value(s) of λ the following homogeneous system of linear equations has a nontrivial solution?

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ -2x_1 & + & \lambda x_2 & + & 2x_3 & = & 0 \\ x_1 & + & 2x_2 & + & \lambda x_3 & = & 0 \end{array}$$

29. Let

$$A = \begin{bmatrix} 2 & 1 & 4 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 3 \\ 2 & 4 & -2 & 8 \end{bmatrix}.$$

a) Use row reduction to calculate $|A|$.
b) Use cofactor expansion to calculate the same determinant.

30. Assuming that the matrix A is invertible, find $B = \text{adj}(\text{adj}(A))$.

31) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Find all 2×2 matrices X such that $XA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

32. Given a 3×3 matrix A and a 4×4 matrix B. If $|A| = 2$ and $|B| = 4$, evaluate the following:
a) $|A^3|$ **b)** $|(2B)^{-1}|$ **c)** $|(AA^T)^{-1}|$ **d)** $|(7B^{-1})^T|$.

33. Let

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 2 & 4 \\ 3 & 6 & 6 \end{bmatrix}, \quad B = AA^T.$$

Find $|A|$, $|B|, |B^T|$ and $|B^{-1}|$.