Exercise Set IV

VECTORS IN 2-SPACE AND 3-SPACE

1. Find an equation for the plane that contains the line

$$x = 2 + t,\ y = -1 - t,\ z = 3t,\ -\infty < t < \infty,$$

and is parallel to the line of intersection of the planes

$$-x + 2y + z = 0 \quad \text{and} \quad x + z + 2 = 0.$$

2. a) Find an equation for the line L passing through the point $P = (1, 0, 1)$ and perpendicular to the plane $x - y + z - 5 = 0$.
b) Where does the line L intersect the plane $x - y + z - 5 = 0$?

3. Show that the lines $x - 3 = 4t,\ y - 4 = t,\ z - 1 = 0,\ t \in \mathbb{R}$ and $\dfrac{x+1}{12} = \dfrac{y-7}{6} = \dfrac{z-5}{3}$ intersect, and find an equation for the plane containing them.

4. Let $P_1 = (1, 0, 0),\ P_2 = (0, 1, 0),\ P_3 = (0, 0, 1),\ P_4 = (1, -1, 6)$.
a) Find an equation for the plane containing the points $P_1,\ P_2$ and P_3.
b) Is there any plane containing the points $P_1,\ P_2,\ P_3$ and P_4.

5. Find an equation for the plane passing through the point P=(1,1,1) and parallel to the line $\dfrac{x-1}{2} = \dfrac{y+2}{4} = z$.

6. Let $\vec{u} = (1, 0, -1),\ \vec{v} = (3, 2, 1),\ \vec{w} = (1, 1, 1)$.
a) Calculate $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$.
b) Find the equation for the plane which contains the following two lines:

$$x = -1 + t,\ y = 1,\ z = 2 - t,\ t \in \mathbb{R} \quad \text{and} \quad x = -1 + 3t,\ y = 1 + 2t,\ z = 2 + t,\ t \in \mathbb{R}.$$

c) Find the volume of the parallelepiped determined by the vectors $\vec{u},\ \vec{v}$ and $\vec{w}$.

7. Show that if $\vec{u}$ is a vector from any point on a line to a point P not on the line, and $\vec{v}$ is a vector parallel to the line, then the distance between P and the line is given by $\dfrac{\|\vec{u} \times \vec{v}\|}{\|\vec{v}\|}$.

8. For the following points in $\mathbb{R}^3$

$$P = (1, 0, -1),\ Q = (2, 4, 5),\ R = (3, 1, 7),$$

use the cross product to find
a) a normal vector to the plane passing through $P,\ Q,\ R$, and the equation of this plane,
b) the area of the triangle $\overset{\triangle}{PQR}$,
c) the distance from the origin to this plane.

9. Find the orthogonal projections $\vec{v_1}$ and $\vec{v_2}$ of $\vec{v} = (1, 4)$ on the directional and the normal vectors of the line $y = \sqrt{2}x$. Check that $\vec{v_1} + \vec{v_2} = \vec{v}$.

10. Let the vectors $\vec{u} = \vec{i} + \vec{k},\ \vec{v} = -\vec{i} + \vec{j},\ \vec{w} = \vec{i} + 2\vec{j} + 3\vec{k}$ be given
a) Use the triple product to verify that the vectors $\vec{u},\ \vec{v},\ \vec{w}$ lie in the same plane.
b) Find the angle between the vectors $\vec{u}$ and $\vec{v}$.

11. Suppose $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^3$ with $\|\vec{u}\| = 2$ and $\|\vec{v}\| = 1$.
a) Find $\|\frac{1}{2}\vec{u} - 3\vec{v}\|$.
b) Find $\|3\vec{u} \times (-5\vec{v})\|$.

12. Find an equation for the plane M that contains the line $x = 2 + t,\ y = -1 - 2t,\ z = 3 - t,\ t \in \mathbb{R}$ and is perpendicular to the plane $x + 3y - z = 0$.

13. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ and give the geometric interpretation of this equation.

14. Determine whether the planes are parallel.
a) $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$.
b) $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$.
c) $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$.

15. Determine whether the line and plane are parallel.
a) $x = -5 - 4t,\ y = 1 - t,\ z = 3 + 2t,\ t \in \mathbb{R}$ and $x + 2y + 3z - 9 = 0$.
b) $x = 3t,\ y = 1 + 2t,\ z = 2 - t,\ t \in \mathbb{R}$, and $4x - y + 2z = 1$.

16. Determine whether the planes are perpendicular.
a) $3x - y + z - 4 = 0$ and $x + 2z = -1$.
b) $x - 2y + 3z = 4$ and $-2x + 5y + 4z = -1$.

17. Determine whether the line and plane are perpendicular.
a) $x = -2 - 4t,\ y = 3 - 2t,\ z = 1 + 2t$ and $2x + y - z = 5$.
b) $x = 2 + t,\ y = 1 - t,\ z = 5 + 3t$ and $6x + 6y - 7 = 0$.

18. Find an equation for the plane that passes through the origin and is parallel to the plane $7x + 4y - 2z + 3 = 0$.

19. Find the point of intersection of the line

$$x - 9 = -5t,\ y + 1 = -t,\ z - 3 = t, \quad -\infty < t < \infty,$$

and the plane $2x - 3y + 4z + 7 = 0$.

20. Find an equation for the plane that passes through $(2, 4, -1)$ and contains the line of intersection of the planes $x - y - 4z = 2$ and $-2x + y + 2z = 3$.

21. Find an equation for the plane through $(-2, 1, 5)$ that is perpendicular to the planes $4x - 2y + 2z = -1$ and $3x + 3y - 6z = 5$.

22. Find an equation for the plane, each of whose points is equidistant from $(-1, -4, -2)$ and $(0, -2, 2)$.

23. Show that the plane whose intercepts with the coordinate axes are $x = a,\ y = b$, and $z = c$ has the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

EUCLIDEAN VECTOR SPACES

1. a) Find the standard matrix for the linear operator T on $\mathbb{R}^3$ which is a composition of a reflection about the $xy-$plane followed by an orthogonal projection on the $xz-$plane followed by multiplication by scalar $\sqrt{2}$.
b) Find all vectors $\vec{u}$ in $\mathbb{R}^3$ such that $T(\vec{u}) = \vec{u}$.

2. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear operator defined by

$$T((x, y, z)) = (y + z, -x - z, -x + y + 4z).$$

Is T one-to-one?

3. The linear operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined as follows:

$$T((x, y, z)) = (-2x - 4y + 3z, 3x + y, 5x + 4y - 2z).$$

a) Is T one-to-one?
b) Find T^{-1} if it exists.

4. Let $\vec{u},\ \vec{v}$ be two vectors in $\mathbb{R}^n$. Show that
a) If $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|$ then $\vec{u}$ and $\vec{v}$ are orthogonal.
b) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$.

5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear operator such that $T((1, 0, 0)) = (1, 0, 1),\ T((0, 1, 0)) = (-1, 1, 2),\ T((0, 0, 1)) = (1, 1, 4)$.
a) Find standard matrix $[T]$ of the transformation T.
b) Is T onto?

6. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a linear operator defined by

$$T((x_1, x_2, x_3, x_4)) = (x_1, x_1 + x_2, 2x_1 + x_2 + x_3, x_3 + x_4)$$

a) Show that T is one-to-one?
b) Find a point $P(x_1, x_2, x_3, x_4) \in \mathbb{R}$ such that $T((x_1, x_2, x_3, x_4)) = (1, 1, 4, 0)$.

7. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Show that T is one-to-one if and only if $\{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = 0\} = \{0\}$.

8. Let $T_1,\ T_2,\ T_3, T_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the following operators on $\mathbb{R}^2$
T_1 : Reflection about the y axis.
T_2 : Counterclockwise rotation through an angle of $30°$.
T_3 : Multiplication by 4.
T_4 : Reflection about the line $y = x$.
a) Find the standard matrix for the operator $T_4(T_3(T_2(T_1)))$
b) Is the operator one-to-one? Explain.

9. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $S : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be transformations. Show that $[ST] = [S][T]$.

10. By inspection, determine the inverse of the given one-to-one linear operator.
a) the reflection about the $x-$axis.
b) the counterclockwise rotation through an angle of $\pi/4$ in $\mathbb{R}^2$
c) multiplication by 3.
d) the reflection about the $yz-$plane in $\mathbb{R}^3$.

11. Determine whether $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation.
a) $T((x, y, z)) = (x, x + y + z)$;
b) $T((x, y, z)) = (1, 1)$.

12. Prove that if $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, then $T(\vec{0}) = \vec{0}$, that is T maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

13. Determine whether multiplication by A is a one-to-one linear transformation.
a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$

14. Use the Cauchy-Schwarz formula to prove that for all real values of $a,\ b$ and θ,

$$(a \cos\theta + b \sin\theta)^2 \leq a^2 + b^2.$$

15. a) Let $\vec{u}$ and $\vec{v}$ be orthogonal vectors in $\mathbb{R}^n$ such that $\|\vec{u}\| = 1$ and $\|\vec{v}\| = 1$. Find $\mathrm{d}(\vec{u}, \vec{v})$.
b) Draw a picture to illustrate this result.