Exercise Set V

EIGENVALUES AND EIGENVECTORS

1. Find the eigenvalues and the corresponding eigenvectors of A^{20}, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

2.Let A and B be $n \times n$ matrices. If λ_1 is an eigenvalue of A, $\vec{v}$ is a corresponding eigenvector of A, and if λ_2 is an eigenvalue of B, $\vec{v}$ is a corresponding eigenvector of B (the same as for A), prove that

a) $\vec{v}$ is an eigenvector of AB.

b) $\vec{v}$ is an eigenvector of $A^5 + B^3$.

3. Let A be a 4×4 matrix. Suppose that the characteristic polynomial of A is

$$p(\lambda) = (\lambda - 1)(\lambda + 3)^2(\lambda - 2).$$

a) What are the eigenvalues of A? What are the eigenvalues of A^3?

b) Is there any nonzero vector $\vec{v} \in \mathbb{R}^4$ such that $A\vec{v} = 5\vec{v}$?

c) Prove that if λ is an eigenvalue of some square matrix B, $\vec{u}$ is a corresponding eigenvector, then $\lambda+7$ is an eigenvalue of $B + 7I$, and $\vec{u}$ is a corresponding eigenvector.

d) Using part **c)**, find all eigenvalues of matrix $A + 7I$.

4. Let A be the matrix below:

$$A = \begin{bmatrix} -5 & -2 \\ 10 & 4 \end{bmatrix}.$$

a) Find the eigenvalues and eigenvectors of A.

b) Diagonalize A, that is, find a matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

c) Calculate the matrix A^{2003}.

5. Find the characteristic equations of the following matrices:

a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, **b)** $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$.

6. Find the eigenvalues and the eigenvectors of the matrices in Exercise 5.

7. Find the eigenvalues and the eigenvectors of the following matrices

a) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, **b)** $\begin{bmatrix} 3 & 0 & -5 \\ 1/5 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$, **c)** $\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$,

d) $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$, **e)** $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$, **f)** $\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$.

8. Find the eigenvalues and the eigenvectors of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

9. Prove: If λ is an eigenvalue of an invertible matrix A and $\vec{v}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\vec{v}$ is a corresponding eigenvector.

10. Prove: If λ is an eigenvalue of A, $\vec{v}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and $\vec{v}$ is a corresponding eigenvector.

11. Find the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}.$$

Then use Exercises 9 and 10 to find the eigenvalues and the eigenvectors of

a) A^{-1}, **b)** $A - 3I$, **c)** $A + 2I$.

12. Prove that if A is a square matrix, then A and A^T have the same eigenvalues.

Hint: Look at the characteristic equation $|A - \lambda I| = 0$.

Remark: While the eigenvalues of A *and* A^T *are the same, corresponding eigenvectors may not be same.*

13. In each part determine whether the matrix is diagonalizable.

a) $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$, **b)** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

14. Find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

15. Determine whether the matrix is diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

16. Compute A^{10}, where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

17. Compute A^{2301}, where

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

18. Find A^n, if n is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$