Exercise Set VI

APPLICATIONS OF EIGENVALUES AND EIGENVECTORS

1. (a) Three companies A, B and C completely control the market for a certain product. Each year A retains 6/10 of its customers while 3/10 switch to B and 1/10 switch to C. Each year B retains 8/10 of its customers while 1/10 switch to A and the same amount switch to C. Each year C holds 7/10 of its customers while 2/10 switch to A and 1/10 switch to B. Let

$$\vec{x}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$$

be the state of the market in year n, with a_n being the market share of A in year n, b_n being the market share of B in year n and c_n being the market share of C in year n.

   i. Find a matrix P such that $\vec{x}_{n+1} = P\vec{x}_n$ for each n.
   ii. Check that the matrix P is stochastic.
   iii. Find the steady state of the process.
   iv. Describe the long-term prospects of each company.

   (b) Prove that any stochastic matrix has the real number 1 as its eigenvalue.

2. Griffins and dragons live in the enchanted forest. The number of dragons D and griffins G each year is determined by their populations in the previous year, according to the formulas:

$$\begin{aligned} D(n) &= 3.5D(n-1) - 3G(n-1) \\ G(n) &= 1.5D(n-1) - G(n-1) \end{aligned}$$

In year 0 there were 3 dragons and 2 griffins.

   (a) Formulate this problem in terms of a matrix linear model.
   (b) How many dragons and griffins there were after one year?
   (c) In the long run, will the populations grow shrink or approach a nonzero equilibrium value?
   (d) After a long time, approximately what will be the ratio of dragons to griffins be?

Do not worry about your answer being factional. Mythical creatures do not have to come in whole units.

3. In a river fish can be classified as juvenile or adult. Let u_n be the number of adult fish in year n and v_n be the number of juvenile fish in year n. It was noticed that from year to year these numbers change according as follows:

$$\begin{aligned} u_n &= 3u_{n-1} - 2v_{n-1} \\ v_n &= u_{n-1} \end{aligned}$$

Suppose that in year 0 there were 1 juvenile fish and 3 adult fishes.

   (a) Describe this process as a linear matrix model, i.e. find the matrix A such that for a sequence of 2-dimensional vectors $\vec{x}_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$ we get $\vec{x}_n = A\vec{x}_{n-1}$. Determine $\vec{x}_0$.
   (b) Find the eigenvalues λ_1, λ_2 of A and the eigenvectors $\vec{v}_1$, $\vec{v}_2$ corresponding to them.
   (c) Express $\vec{x}_0$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, i.e. find a_1 and a_2 such that $\vec{x}_0 = a_1\vec{v}_1 + a_1\vec{v}_2$ and find $\vec{x}_n$ as $a_1\lambda_1^n\vec{v}_1 + a_2\lambda_2^n\vec{v}_2$.
   (d) Give the numbers of juvenile and adult fish at year n.

4. (a) Diagonalize the matrix A from the previous question, i.e. find an invertible matrix P and the diagonal matrix D such that $P^{-1}AP = D$.
   (b) Calculate A^n.
   (c) Use (b) to calculate $\vec{x}_n$ from the previous question as $A\vec{x}_0$. (You must get the same answer as in the previous question).

5. An animal herd we may consider that the cohort C_1 (newborn) contains animals of ages 0 to 1, the cohort C_2 (juveniles) contains animals of ages 1 to 2, the cohort C_3 (adults) contains animals of ages 2 and above. Only adults reproduce, at the rate 0.40 of newborn per adult per year. We assume that 65% of the newborn survive to become juveniles, 75% of juveniles survive to adults, and 95% of adults survive to lie another year.

   (a) Denote the number of newborn at year n as a_n, the number of juveniles at year n as b_n, and the number of adults at year n as c_n, and write down the equations for these numbers.

   (b) Convert these into a linear model.

   (c) Suppose that Maple tells you that there are three distinct eigenvalues for the matrix A with $\lambda_1 = 1.108651595$, $|\lambda_2| < 1$, $|\lambda_3| < 1$ and $\vec{v_1} = \begin{bmatrix} 0,3564803352 & 0,2090036394 & 0.9880312278 \end{bmatrix}^T$. At which speed this herd grows and what is the proportion of newborn, juvenile and adults in it. You may assume that this herd have lived for quite some time.

6. I tend to be rather moody at times. If I am in a good mood today, there is 80% chance I'll still be in a good mood tomorrow. But if I'm grumpy today, there is only a 60% chance that my mood will be good tomorrow.

   (a) Describe the transition matrix for this situation.

   (b) If I am in a good mood today, what is the probability that I will be in a good mood one week from now (I will be preparing exam questions on that day!)

   (c) If I am in a good mood today, what is the probability that I will be in a good mood two weeks from now (I will be grading!)

   (d) Over the long term, what percentage of the time am I in a good mood? (You may inform my future students!)

7. A car rental agency in Chicago has offices at two airports, O'Hare and Midway, as well as one downtown in the Loop. The table below shows the probabilities that a car rented at one particular location will be returned to some other location. Assuming that all cars are returned by the end of the day and that the agency does not move cars itself from office to office, what fractions of its total fleet, on the average, will be available at each of its offices?

Rentals	**Returns**		
	O'Hare	Midway	Loop
O'Hare	0.7	0.1	0.2
Midway	0.2	0.6	0.2
Loop	0.5	0.3	0.2

8. A certain metropolitan area has a population of 9 million, of whom 5 million live in the city, and 4 million live in the suburbs. Each year, only 20% of the suburbanites move to the city (and 80% stay in the suburbs), while 40% of the city dwellers move to the suburbs (and 60% stay in the city). Set up the time evolution equations and the diagonalize the coefficient matrix. (You may ignore the effects of births, deaths, and migration into and out of the metropolitan area.) In the long run, will the population of the city decrease to zero or stabilize? In the previous example, suppose that the initial populations were 9 million suburbanites and 3 million city dwellers. How would that affect the long-term distribution?

9. A very different football team gets overconfident after a win but works hard after a loss. After winning a game, it has only a 20% change of winning the next game, but after losing a game it has a 90% change of winning the next game. In the long run, what fraction of the games will this contrary team win?

REAL VECTOR SPACES

1. Let W_1 be a set of all vectors (a, b, c, d) in $\mathbb{R}^4$ such that $a + d = 0$ and W_2 be a set of all vectors (a, b, c, d) in $\mathbb{R}^4$ such that $ad = 0$. Is W_1 a subspace of $\mathbb{R}^4$? Is W_2 a subspace of $\mathbb{R}^4$?

2. Let $\vec{u_1} = (1, 1, 0)$, $\vec{u_2} = (1, 0, 1)$, $\vec{u_3} = (0, 1, 1)$

   (a) Show that $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is linearly independent.

   (b) Express $\vec{v} = (1, 2, 3)$ as a linear combination of $\vec{u_1}, \vec{u_2}, \vec{u_3}$.

3. (a) Give an example of a system of three vectors $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ in $\mathbb{R}^3$ with the following properties:
   i. $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ is linearly dependent.
   ii. Each of the three systems $\{\vec{u_1}, \vec{u_2}\}$, $\{\vec{u_1}, \vec{u_3}\}$, $\{\vec{u_2}, \vec{u_3}\}$ is linearly independent.

   Give reasons for your answer and the geometric interpretation.

   (b) Let A and B are arbitrary 2×2 matrices. Prove that the set W of 2×2 matrices X such that $AX = X^T B$ is a subspace of the space V of all real 2×2 matrices. (Hint: $(X+Y)^T + X^T + Y^T$.)

   (c) Find a basis for the space W from part (b), when

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

4. (a) For which real value of λ do the following vectors

$$\vec{v_1} = (\lambda, 1, 1),\ \vec{v_2} = (1, \lambda, 1,),\ \vec{v_3} = (0, 3, 1)$$

   form a linearly dependent set in $\mathbb{R}^3$?

   (b) Prove that the set W of all 2×2 matrices of the form

$$\begin{bmatrix} a & a-b \\ a+2b & b \end{bmatrix}, \quad a, b \in \mathbb{R}$$

   is a subspace of the vector space V of all 2×2 matrices.

5. Let W be the set of all vectors in $\mathbb{R}^4$ such that their first coordinate is twice their third coordinate and that the sum of their second and fourth coordinates is zero.

   (a) Prove that W is subspace in $\mathbb{R}^4$.

   (b) Write down a homogeneous system of linear equations for which W is the set of solutions.

   (c) Represent W as a span of a system of vectors.

6. Let $V = \mathbb{R}^3$. Let $\vec{v}_1 = (1, 1, 0)$, $\vec{v}_2 = (1, 0, 1)$, $\vec{v}_3 = (0, 1, 0)$, $\vec{v}_4 = (4, -1, 6)$ and $\vec{v}_5 = (0, 0, 0)$ be vectors in V. Determine whether each of the following sets form a basis for V. Explain.

   (a) $S_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

   (b) $S_2 = \{\vec{v}_1, \vec{v}_2, \vec{v}_5\}$.

   (c) $S_3 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

7. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 8 & 4 & 2 & 0 \\ -2 & 1 & 1 & -4 \end{bmatrix}.$$

   (a) Find a basis for the row space of A.

   (b) What is rank(A), nullity(A)?

8. Given the vector space V and the subset $W \subset V$. Determine whether W is a subspace of V in the following cases. Explain.

   (a) $V = \mathbb{R}^3$, $W = \{(x, y, z) \in \mathbb{R}^3 : \ (x, y-2, z) \cdot (1, 4, 2) = 0\}$.

   (b) $V = M_{2\times 2}$, $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c = 0 \right\}$.

   Note: $M_{2\times 2}$ is the set of all 2×2 matrices with usual matrix addition and scalar multiplication by reals.

9. Given

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

   (a) Show that $S = \{A, B, C, D\}$ is a basis for $M_{2\times 2}$.

(b) Represent the matrix

$$E = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

as a linear combination of the matrices from S.

10. Suppose that $S = \{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set and that the vector $\vec{v}_3$ does not belong to $\text{span}(S)$. Prove that $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also a linearly independent set.

11. Determine bases for the following subspaces of $\mathbb{R}^3$.

    (a) The plane $5x + 3y - 7z = 0$.

    (b) The line $x = 8t,\ y = -6t,\ z = 101t,\ t \in \mathbb{R}$.

12. Find bases for the row space of A, column space of A. Find $\text{rank}(A)$ and $\text{nullity}(A)$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix}.$$

13. In each part, write down a matrix with required property or explain why no such matrix exit.

    (a) Column space contains $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}$.

    (b) Column space $= \mathbb{R}^2$, row space $= \mathbb{R}^4$.

    (c) Column space contains $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, rank=3.

    (d) Column space has basis $\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, null space has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

14. In each part a set of objects is given together with operations of addition ($\oplus$) and scalar multiplication ($\odot$). Determine which set(s) are vector spaces under the given operations. Justify your answer.

    (a) The set of all triples of real numbers (x, y, z) with the operations

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z') \quad \text{and} \quad k \odot (x, y, z) = (kx, y, z).$$

    (b) The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) \oplus (1, y') = (1, y + y') \quad \text{and} \quad k \odot (1, y) = (1, ky).$$

15. Determine which of the following are subspaces of M_{22}.

    (a) all 2×2 matrices with integer entries.

    (b) all matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

    where $a + b + c + d = 0$.

    (c) all 2×2 matrices A such that $|A| = 0$.

    (d) all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

16. Determine which of the following are subspaces of M_{nn}.

    (a) all $n \times n$ matrices A such that $\text{tr}(A) = 0$.

    (b) all $n \times n$ matrices A such that $A^T = -A$.

    (c) all $n \times n$ matrices A such that $|A| \neq 0$.

    (d) all $n \times n$ matrices A such that $AB = BA$ for a fixed $n \times n$ matrix B.

17. In each part determine whether the given vectors span $\mathbb{R}^3$.

   (a) $\vec{v}_1 = (3, 1, 4),\ \vec{v}_2 = (2, -3, 5),\ \vec{v}_3 = (5, -2, 9),\ \vec{v}_4 = (1, 4, -1)$.
   (b) $\vec{v}_1 = (1, 2, 6),\ \vec{v}_2 = (3, 4, 1),\ \vec{v}_3 = (4, 3, 1),\ \vec{v}_4 = (3, 3, 1)$.

18. (a) Show that the vectors $\vec{v}_1 = (0, 3, 1, -1),\ \vec{v}_2 = (6, 0, 5, 1),\ \vec{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in $\mathbb{R}^4$.
   (b) Express each vector as a linear combination of the other two.

19. Prove: For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, the vectors $\vec{u} - \vec{v}$, $\vec{v} - \vec{w}$ and $\vec{w} - \vec{u}$ form a linearly dependent set.

20. Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be linearly independent. Prove that $\vec{u} + \vec{v}$, $\vec{v} + \vec{w}$, $\vec{w} + \vec{u}$ are also linearly independent. Compare this result with the previous problem. Why are the determinants

$$\begin{vmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

   relevant.

21. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly dependent?

   (a) $(3, 8, 7, -3),\ (1, 5, 3, -1),\ (2, -1, 2, 6),\ (1, 4, 0, 3)$.
   (b) $(0, 0, 2, 2),\ (3, 3, 0, 0),\ (1, 1, 0, -1)$.
   (c) $(0, 3, -3, -6),\ (-2, 0, 0, -6),\ (0, -4, -2, -2), (0, -8, 4, -4)$.
   (d) $(3, 0, -3, 6),\ (0, 2, 3, 1),\ (0, -2, -2, 0), (-2, 1, 2, 1)$.

22. Determine bases for the following subspaces of $\mathbb{R}^3$.

   (a) The plane $3x - 2y + 5z = 0$.
   (b) The plane $x - y = 0$.
   (c) The line $x = 2t,\ y = -t,\ z = 4t$.
   (d) All vectors of the form (a, b, c), where $b = a + c$.

23. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

   (a) $(1, 0, 0),\ (2, 2, 0),\ (3, 3, 3)$.
   (b) $(3, 1, -4),\ (2, 5, 6),\ (1, 4, 8)$.
   (c) $(2, -3, 1),\ (4, 1, 1),\ (0, -7, 1)$.
   (d) $(1, 6, 4),\ (2, 4, -1),\ (-1, 2, 5)$.

24. Determine the dimensions of the following subspaces of $\mathbb{R}^4$.

   (a) All vectors of the form $(a, b, c, 0)$.
   (b) All vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$.
   (c) All vectors of the form (a, b, c, d) where $a = b = c = d$.

25. Find a basis for the null space, row space and column space of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}.$$

26. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors.

   (a) $(1, 1, -4, -3),\ (2, 0, 2, -2),\ (2, -1, 3, 2)$.
   (b) $(-1, 1, -2, 0),\ (3, 3, 6, 0),\ (9, 0, 0, 3)$.
   (c) $(1, 1, 0, 0),\ (0, 0, 1, 1),\ (-2, 0, 2, 2),\ (0, -3, 0, 3)$.

27. Find the rank and nullity of the matrix A.

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}.$$

28. In each part use the information in the table to find the dimension of the row space of A, column space of A, the null space of A.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
Rank(A)	3	2	1	2	2	0	2

29. Discuss how the rank of A varies with t.

(a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}.$

(b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}.$

30. **Theorem.** If $\vec{x}_0$ denotes any single solution of a consistent linear system $A\vec{x} = \vec{b}$, and if vectors $\{\vec{v}_1,\ \vec{v}_2, \cdots, \vec{v}_k\}$ form a basis for the null space of A, that is, the solution space of the homogeneous system $A\vec{x} = \vec{0}$, then every solution of $A\vec{x} = \vec{b}$ can be expressed in the form

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \qquad (1)$$

and, conversely, for all choices of scalars $c_1,\ c_2, \cdots, c_k$, the vector $\vec{x}$ in this formula is a solution of $A\vec{x} = \vec{b}$.

Definition. In formula (1), the vector $\vec{x}_0$ is called a **particular solution** of $A\vec{x} = \vec{b}$ and the expression

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k, \quad c_1,\ c_2, \cdots, c_k \in \mathbb{R}$$

is called the **general solution** of $A\vec{x} = \vec{b}$.

Using the above information, find the general solution of the system

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6 \end{array}$$

31. Suppose that $x_1 = -1,\ x_2 = 2,\ x_3 = 4,\ x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas

$$x_1 = -3r + 4s,\ x_2 = r - s,\ x_3 = r,\ x_4 = s, \quad r, s \in \mathbb{R}.$$

(a) Find the general solution of $A\vec{x} = \vec{0}$.

(b) Find the general solution of $A\vec{x} = \vec{b}$.