ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 205 - Basic Linear Algebra

2018-2019 Spring Semester

Final Examination

27.05.2018, 15:00

ANSWER KEY

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

Question	Grade	Out of
1		20
2		20
3		30
4		30
Total		100

IMPORTANT NOTES:

1) Check that the exam paper contains 4 questions.

2) Show all steps of your work. Both the correct method and correct result are necessary to get full point.

3) Calculators and cell phones are <u>NOT ALLOWED</u>.

4) It is not allowed to leave the exam during the first 30 minutes.

1) (20 points) Find a basis for $\mathbb{P}_2$ which contains the set of polynomials $\{t^2+1, t-1\}$.

Complete the given set to $\{t^2+1, t-1, \underbrace{t^2, t, 1}_{\text{standart basis for } \mathbb{P}_2}\} \xrightarrow[\text{isomorphic to}]{\text{which is}} \left\{ \underset{v_1}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}0\\1\\-1\end{bmatrix}}, \underbrace{\underset{v_3}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{v_5}{\begin{bmatrix}0\\0\\1\end{bmatrix}}}_{\text{standart basis for } \mathbb{R}^3} \right\}$

then,

$$\begin{array}{c}\begin{bmatrix}1&0&1&0&0\\0&1&0&1&0\\1&-1&0&0&1\end{bmatrix}\\ \begin{matrix}v_1&v_2&v_3&v_4&v_5\end{matrix}\end{array} \xrightarrow{-R_1+R_3\to R_3} \begin{bmatrix}1&0&1&0&0\\0&1&0&1&0\\0&-1&-1&0&1\end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{array}{ccccc}\textcircled{1}&0&1&0&0\\0&\textcircled{1}&0&1&0\\0&0&-1&1&\textcircled{1}\\ \uparrow&\uparrow&&&\uparrow\\ v_1&v_2&&&v_5\end{array}$$

$\{v_1, v_2, v_5\}$ equivalently, $\{t^2+1, t-1, 1\}$ form a basis for $\mathbb{P}_2$.

OR

$$\xrightarrow[\substack{R_3+R_1\to R_1\\ -R_3\to R_3}]{} \begin{array}{ccccc}1&0&0&1&1\\0&1&0&1&0\\0&0&1&-1&-1\\ \uparrow&\uparrow&\uparrow&&\\ v_1&v_2&v_3&&\end{array}$$

OR $\{v_1, v_2, v_3\}$

$\{t^2+1, t-1, t^2\}$

2) For the given matrix $A = \begin{bmatrix}1&0&-1&2\\8&4&2&0\\-2&1&1&-4\end{bmatrix}$,

a) (12 points) find a basis for the rowspace of A,
b) (8 points) calculate $rank(A)$ and $nullity(A)$.

a) $A = \begin{bmatrix}1&0&-1&2\\8&4&2&0\\-2&1&1&-4\end{bmatrix} \xrightarrow{\text{is reduced to}} \begin{bmatrix}\textcircled{1}&0&0&6/7\\0&\textcircled{1}&0&-8/7\\0&0&\textcircled{1}&-8/7\end{bmatrix}$

so, basis for row space of A is $\{[1\ 0\ 0\ 6/7], [0\ 1\ 0\ -8/7], [0\ 0\ 1\ -8/7]\}$

OR

originally $\{[1\ 0\ -1\ 2], [8\ 4\ 2\ 0], [-2\ 1\ 1\ -4]\}$

b)

$\Rightarrow rank(A) = \textcircled{3}$,

$\Rightarrow$ Since $n=4$ (3×4 matrix) then $nullity(A) = 4-3 = \textcircled{1}$

3) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+2c \\ a-b \\ 2a+b \end{bmatrix}$.

a) (5 points) Show that L is linear.

b) (14 points) Prove that L is invertible.

c) (5 points) Calculate the standart matrix representing L.

d) (6 points) Find $L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

a) Linearity: 1) $L(u+v) = L\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix}\right) = \begin{bmatrix} (a_1+a_2)+(b_1+b_2)+2(c_1+c_2) \\ (a_1+a_2)-(b_1+b_2) \\ 2(a_1+a_2)+(b_1+b_2) \end{bmatrix}$

$$= \begin{bmatrix} a_1+b_1+2c_1 \\ a_1-b_1 \\ 2a_1+b_1 \end{bmatrix} + \begin{bmatrix} a_2+b_2+2c_2 \\ a_2-b_2 \\ 2a_2+b_2 \end{bmatrix} = L(u)+L(v) \checkmark$$

2) $L(k.u) = L\left(\begin{bmatrix} ka_1 \\ kb_1 \\ kc_1 \end{bmatrix}\right) = \begin{bmatrix} ka_1+kb_1+2kc_1 \\ ka_1-kb_1 \\ 2ka_1+kb_1 \end{bmatrix}$

$$= k\begin{bmatrix} a_1+b_1+2c_1 \\ a_1-b_1 \\ 2a_1+b_1 \end{bmatrix} = k.L(u) \checkmark$$

L is a linear transformation.

b) $\ker(L) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} x+y+2z \\ x-z \\ 2x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \left.\begin{matrix} x+y+2z=0 \\ x-z=0 \\ 2x+y=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=0 \\ z=0 \end{matrix}$ (trivial soln.)

so $\ker(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = 0_{\mathbb{R}^3}$, nullity$(L)=0 \Longrightarrow$ rank$(L) = 3-0 = \boxed{3}$

$\operatorname{Ker}(L) = 0_{\mathbb{R}^3} \underset{\text{by Thm}}{\Longrightarrow} L$ is (1-1)

L is (1-1) & $\dim \mathbb{R}^3 = \dim \mathbb{R}^3 \xrightarrow[\text{by Thm.}]{} L$ is onto.

Since L is (1-1) & onto then L^{-1} exist. (L is invertible)

c) $L(x) = A.x$ where $A = [L(e_1)\ L(e_2)\ L(e_3)]$, e_1, e_2, e_3 are standart basis of $\mathbb{R}^3$.

$$A = \left[L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right]$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} //$$

standart matrix representing L.

d) $L^{-1}(x) = A^{-1}x$ $\quad [A|I]$ is reduced to $\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -2 & -4 & 3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$ (with $\underbrace{}_{I}$ $\underbrace{}_{A^{-1}}$)

$$L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2b-c \\ -2a-4b+3c \\ a+b-c \end{bmatrix} //$$

4) For the vector space $\mathbb{R}^3$ with the inner product $\langle a, b\rangle = a_1b_1 + 2a_2b_2 + 3a_3b_3$.

a) (20 points) Apply Gram-Schmidt method to transform the basis $S = \{(2, -1, 1), (3, 0, 1), (4, 2, -1)\}$ into an orthonormal basis T.

b)(10 points) Compute the coordinate vector $[v]_T$ for the given $v = (2, 0, 2)$.

a) $S=\{(2,-1,1),(3,0,1),(4,2,-1)\} = \{u_1,u_2,u_3\}$

Applying Gram-Schmidt to $\{u_1,u_2,u_3\}$:

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$$\vec{V}_1 = \vec{U}_1 = \begin{bmatrix}2\\-1\\1\end{bmatrix} //$$

$$\vec{V}_2 = \vec{U}_2 - \frac{\langle \vec{U}_2, \vec{V}_1\rangle}{\langle \vec{V}_1, \vec{V}_1\rangle}\vec{V}_1 = \begin{bmatrix}3\\0\\1\end{bmatrix} - \frac{\left\langle \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\rangle}{\left\langle \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\rangle}\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} //$$

$$\vec{V}_3 = \vec{U}_3 - \frac{\langle \vec{U}_3, \vec{V}_1\rangle}{\langle \vec{V}_1, \vec{V}_1\rangle}\vec{V}_1 - \frac{\langle \vec{U}_3, \vec{V}_2\rangle}{\langle \vec{V}_2, \vec{V}_2\rangle}\vec{V}_2 \quad 9$$

$$= \begin{bmatrix}4\\2\\-1\end{bmatrix} - \frac{1}{9}\begin{bmatrix}2\\-1\\1\end{bmatrix} - \frac{8}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}10/9\\-5/9\\-10/9\end{bmatrix} //$$

So $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is the orthogonal set.

$$W_1 = \frac{V_1}{\|V_1\|} = \frac{V_1}{\langle V_1, V_1\rangle^{1/2}} = \frac{1}{3}\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$W_2 = \frac{V_2}{\|V_2\|} = \frac{V_2}{\langle V_2, V_2\rangle^{1/2}} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$W_3 = \frac{V_3}{\|V_3\|} = \frac{1}{\frac{5\sqrt{2}}{3}}\begin{bmatrix}10/9\\-5/9\\-10/9\end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix}2\\-1\\-2\end{bmatrix}$$

$\{W_1, W_2, W_3\}$ becomes orthonormal basis//

//

b) $V = C_1W_1 + C_2W_2 + C_3W_3$ then $[V]_T = \begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}$

So
$$\begin{bmatrix}2\\0\\2\end{bmatrix} = C_1\begin{bmatrix}2/3\\-1/3\\1/3\end{bmatrix} + C_2\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\end{bmatrix} + C_3\begin{bmatrix}2/3\sqrt{2}\\-1/3\sqrt{2}\\-2/3\sqrt{2}\end{bmatrix} \Longrightarrow [V]_T = \begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix} = \begin{bmatrix}10/3\\\frac{2\sqrt{3}}{3}\\\frac{-4\sqrt{2}}{3}\end{bmatrix} //$$