ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 205 - Basic Linear Algebra

2019-2020 Fall

Final Examination
06.01.2020, 12:30

KEY

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

Question	Grade	Out of
1		30
2		25
3		25
4		30
Total		110

IMPORTANT NOTES:

1) Check that the exam paper contains 4 questions.

2) Show all steps of your work. Both the correct method and correct result are necessary to get full point.

3) Calculators and cell phones are NOT ALLOWED.

4) It is not allowed to leave the exam during the first 30 minutes.

1) In $\mathbb{R}^4$, let W be the subset of all vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{R}^4$ satisfying $a_4 - a_3 = a_2 - a_1$.

a) (10 points) Show that W is a subspace of $\mathbb{R}^4$.

b) (10 points) Show that $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans W.

c) (10 points) Find a subset of S that is a basis for W and determine its dimension.

a) $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a_4 - a_3 = a_2 - a_1 \right\}$

Since $\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\in W} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_3 + a_2 - a_1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$,

$W = \text{span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{u_3} \right\}$. Thus, W is a subspace of $\mathbb{R}^4$.

call u_1 u_2 u_3

b) Since $S = \{ u_1, u_2, u_1 + u_2 + u_3, u_3 \}$ then $W = \text{span}\, S$.
(S is spanned by W)

c) Choosing linearly independent vectors in S,
clearly $\{u_1, u_2, u_3\}$ form a basis for W.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ //

2) Define on $\mathbb{R}^3$ the inner product $\langle u, v\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$ for any vector $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Let $S = \{(1,0,0), (1,1,0), (1,1,1)\}$ be a subset of $\mathbb{R}^3$.

a) (10 points) Show that S is a basis for $\mathbb{R}^3$.
b) (15 points) Starting from S find an orthonormal basis for $\mathbb{R}^3$ using Gram-Schmidt orthogonalization process.

a) $S = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis for $\mathbb{R}^3$ iff

1) S spans $\mathbb{R}^3$: $\exists c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1(1,0,0)+c_2(1,1,0)+c_3(1,1,1)=(a,b,c)$ for any $(a,b,c)\in\mathbb{R}^3$

$$\Rightarrow \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}a-b\\b-c\\c\end{bmatrix}$$

is a soln. set having inf. many solns. for the parameters $a,b,c\in\mathbb{R}$.

So, S spans $\mathbb{R}^3$.

2) S is lin. indep. : Since $\det\left(\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\right)=1\neq 0$ then S is a lin. indep. set.

Hence, S form a basis for $\mathbb{R}^3$.

b) $S = \{\alpha_1, \alpha_2, \alpha_3\} = \{(1,0,0),(1,1,0),(1,1,1)\}$ (basis for $\mathbb{R}^3$)

Applying Gram-Schmidt method:

$$\beta_1 = \alpha_1 = \boxed{(1,0,0)}$$

$$\beta_2 = \alpha_2 - \frac{\langle\alpha_2,\beta_1\rangle}{\langle\beta_1,\beta_1\rangle}\beta_1 = (1,1,0) - \frac{\langle(1,1,0),(1,0,0)\rangle}{\langle(1,0,0),(1,0,0)\rangle}(1,0,0)$$

$$= (1,1,0) - \frac{1}{1}(1,0,0) = \boxed{(0,1,0)}$$

$$\beta_3 = \alpha_3 - \frac{\langle\alpha_3,\beta_1\rangle}{\langle\beta_1,\beta_1\rangle}\beta_1 - \frac{\langle\alpha_3,\beta_2\rangle}{\langle\beta_2,\beta_2\rangle}\beta_2$$

$$= (1,1,1) - \frac{1}{1}(1,0,0) - \frac{2}{2}(0,1,0) = \boxed{(0,0,1)}$$

So, we obtain an orthogonal basis $\boxed{\{\beta_1,\beta_2,\beta_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}}$

By orthonormalization process, we finally get

$$\gamma_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{1}{1}(1,0,0) = (1,0,0)$$

$$\gamma_2 = \frac{\beta_2}{\|\beta_2\|} = \frac{1}{\sqrt{2}}(0,1,0) = (0,\tfrac{1}{\sqrt{2}},0)$$

$$\gamma_3 = \frac{\beta_3}{\|\beta_3\|} = \frac{1}{\sqrt{3}}(0,0,1) = (0,0,\tfrac{1}{\sqrt{3}})$$

Hence, $\boxed{\{\gamma_1,\gamma_2,\gamma_3\} = \{(1,0,0),(0,\tfrac{1}{\sqrt{2}},0),(0,0,\tfrac{1}{\sqrt{3}})\}}$ is an orthonormal basis for $\mathbb{R}^3$

3) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L((a_1, a_2)) = (a_1, a_1 + a_2, a_2)$.

a) (5 points) Find a basis for $KerL$.

b) (8 points) Find a basis for $RangeL$.

c) (8 points) Determine *nullity(L)* and *rank(L)*.

d) (4 points) Is L one-to-one? Explain.

a) $Ker L = \{ (a_1, a_2) \in \mathbb{R}^2 \mid L((a_1,a_2)) = 0_{\mathbb{R}^3} = (0,0,0) \}$

So, $L((a_1,a_2)) = (a_1, a_1+a_2, a_2) = (0,0,0) \Rightarrow \left.\begin{array}{l} a_1=0 \\ a_2=0 \\ a_1+a_2=0 \end{array}\right\}$ So, $\boxed{a_1=a_2=0}$ $(a_1,a_2)=(0,0)$

Hence, $\boxed{Ker L = \{0_{\mathbb{R}^2}\}}$ $\Rightarrow$ No basis for Ker L!

b) $L((a_1,a_2)) = (a_1, a_1+a_2, a_2) = (b_1, b_2, b_3) \in \mathbb{R}^3$

$\Rightarrow b_2 = b_1 + b_3$

So the image of $\forall (a_1,a_2) \in \mathbb{R}^2$ is $\underbrace{\text{the set of vectors } (b_1,b_2,b_3) \in \mathbb{R}^3 \text{ such that } b_2 = b_1+b_3}_{Range L}$.

$(b_1,b_2,b_3) = (b_1, b_1+b_3, b_3)$
$= b_1(1,1,0) + b_3(0,1,1)$ for any $b_1, b_3 \in \mathbb{R}$.

Hence, $\boxed{\{(1,1,0), (0,1,1)\}}$ form a basis for Range L.

c) $rank(L) = \dim(Range L) = 2$
$nullity(L) = \dim(Ker L) = 0$

d) L is (1-1) since $Ker L = \{0_{\mathbb{R}^2}\}$. ✓

4) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be ordered basis for the vector space $\mathbb{R}^2$.

a) (5 points) <u>Find the coordinate vector</u> $[v]_S$ where $v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

b) (12 points) <u>Calculate the transition matrix</u> $P_{T \leftarrow S}$.

c) (5 points) <u>Find</u> $[v]_T$ directly for $v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

d) (8 points) <u>Find</u> $[v]_T$ using the matrix obtained in *part (b)*.

a) $V = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{matrix} c_1 + c_2 = 5 \\ -3c_2 = 1 \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} c_2 = -1/3 \\ c_1 = 16/3 \end{matrix}}$

basis of S

So $[V]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 16/3 \\ -1/3 \end{bmatrix}$ // coordinate vector w.r.t. S.

b) $\left[\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -3 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right] \xrightarrow[-R_2+R_1 \to R_1 \; (2)]{\frac{1}{2}R_2 \to R_2 \; (1)} \left[\begin{array}{cc|c|c} 1 & 0 & 1/2 & 2 \\ 0 & 1 & 1/2 & -1 \end{array}\right]$

$\underbrace{}_{I}$ $\underbrace{}_{P_{T \leftarrow S}}$ transition matrix from S basis to T basis

So, $P_{T \leftarrow S} = \begin{bmatrix} 1/2 & 2 \\ 1/2 & -1 \end{bmatrix}$ //

c) $V = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{matrix} c_1 + c_2 = 5 \\ -c_1 + c_2 = 1 \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} c_2 = 3 \\ c_1 = 2 \end{matrix}}$

basis of T

So, $[V]_T = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ // coordinate vector w.r.t. T.

d) $\boxed{[V]_T = P_{T \leftarrow S} \cdot [V]_S}$

$= \begin{bmatrix} 1/2 & 2 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} 16/3 \\ -1/3 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} // = [V]_T$