ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 205 - Basic Linear Algebra

2019-2020 Fall

Midterm Examination
18.11.2019, 17:30

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 90 minutes

Answer key

Question	Grade	Out of
1		30
2		30
3		15
4		15
5		15
Total		105

IMPORTANT NOTES:

1) Check that the exam paper contains 5 questions.

2) Show all steps of your work. Both the correct method and correct result are necessary to get full point.

3) Calculators and cell phones are <u>NOT ALLOWED</u>.

4) It is not allowed to leave the exam during the first 30 minutes.

1) (30 points) Use Gauss-Jordan elimination method to solve the linear system

$$
\begin{aligned}
x_1 + 2x_3 + x_4 &= 4\\
x_1 - x_2 + 2x_4 &= 12\\
2x_1 + x_2 + x_4 &= 12\\
x_1 + 2x_2 + x_3 + x_4 &= 12\\
3x_1 + 3x_2 + x_3 + 2x_4 &= 24.
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 1 & 4\\
1 & -1 & 0 & 2 & 12\\
2 & 1 & 0 & 1 & 12\\
1 & 2 & 1 & 1 & 12\\
3 & 3 & 1 & 2 & 24
\end{array}\right]
\xrightarrow[\substack{R_4 - R_1 \to R_4\\ R_5 - 3R_1 \to R_5}]{\substack{R_1 - R_2 \to R_2\\ R_3 - 2R_1 \to R_3}}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 1 & 4\\
0 & 1 & 2 & -1 & -8\\
0 & 1 & -4 & -1 & 4\\
0 & 2 & -1 & 0 & 8\\
0 & 3 & -5 & -1 & 12
\end{array}\right]
\xrightarrow[\substack{R_4 - 2R_2 \to R_4\\ R_5 - 3R_2 \to R_5}]{R_3 - R_2 \to R_3}
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 1 & 4\\
0 & 1 & 2 & -1 & -8\\
0 & 0 & -6 & 0 & 12\\
0 & 0 & -5 & 2 & 24\\
0 & 0 & -11 & 2 & 36
\end{array}\right]
$$

$$
\xrightarrow[\substack{R_5 - \frac{11}{6}R_3 \to R_4\\ -\frac{1}{6}R_3}]{R_4 - \frac{5}{6}R_3 \to R_4}
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 1 & 4\\
0 & 1 & 2 & -1 & -8\\
0 & 0 & 1 & 0 & -2\\
0 & 0 & 0 & 2 & 14\\
0 & 0 & 0 & 2 & 14
\end{array}\right]
\xrightarrow[\frac{1}{2}R_4 \to R_4]{R_5 - R_4 \to R_5}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 1 & 4\\
0 & 1 & 2 & -1 & -8\\
0 & 0 & 1 & 0 & -2\\
0 & 0 & 0 & 1 & 7\\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\xrightarrow[R_1 + R_4 \to R_1]{R_1 - R_4 \to R_1}
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 0 & -3\\
0 & 1 & 2 & 0 & -1\\
0 & 0 & 1 & 0 & -2\\
0 & 0 & 0 & 1 & 7\\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\xrightarrow[R_2 - 2R_3 \to R_2]{R_1 - 2R_3 \to R_1}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 3\\
0 & 0 & 1 & 0 & -2\\
0 & 0 & 0 & 1 & 7\\
0 & 0 & 0 & 0 &
\end{array}\right]
$$

$x_1 = 1$

$x_2 = 3$

$x_3 = -2$

$x_4 = 7$

unique soln.

$$
X = \begin{bmatrix} 1\\ 3\\ -2\\ 7 \end{bmatrix}
$$

is the unique soln. vector.

2) (30 points) Find the values of the real parameters α and β for which the following linear system

$$\begin{aligned} x_1 - x_2 + 2\alpha x_3 + x_4 &= \beta \\ x_1 - 2x_2 + 2\alpha x_3 + \alpha x_4 &= 1+\beta \\ 2x_1 - 2x_2 + 5\alpha x_3 + 2x_4 &= 1+\beta \\ 3x_1 - 3x_2 + 6\alpha x_3 + 3\alpha x_4 &= 1 \end{aligned}$$

has
a) a unique solution,
b) no solution,
c) infinitely many solutions.
Explain clearly your answer.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2\alpha & 1 & \beta \\ 1 & -2 & 2\alpha & \alpha & 1+\beta \\ 2 & -2 & 5\alpha & 2 & 1+\beta \\ 3 & -3 & 6\alpha & 3\alpha & 1 \end{array}\right] \xrightarrow[\substack{R_3 - 2R_1 \to R_3 \\ R_4 - 3R_1 \to R_4}]{\substack{-R_1 + R_2 \to R_2 \\ -R_2 \to R_2}} \left[\begin{array}{cccc|c} 1 & -1 & 2\alpha & 1 & \beta \\ 0 & 1 & 0 & 1-\alpha & -1 \\ 0 & 0 & \alpha & 0 & 1-\beta \\ 0 & 0 & 0 & 3\alpha-3 & 1-3\beta \end{array}\right]$$

i) $\alpha \neq 0, \alpha \neq 1$ there is a unique solucion

ii) $\alpha = 0, \ \beta = 1$ infinity many solution
$\alpha = 1 \quad \beta = \frac{1}{3}$ " " "

iii) $\alpha = 0, \ \beta \neq 1$ No solution.
$\alpha = 1 \ ; \ \beta \neq 1/3$ "

3) (15 points) Find the value of $a \in \mathbb{R}$ for which the matrix $A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is invertible,

then solve the corresponding system $Ax = b$ by using inverse matrix where $b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$.

$\boxed{\det A = 1}$ so A is invertible $\forall a \in \mathbb{R}$

$$\left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - aR_4 \to R_3} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - aR_3 \to R_2} \left[\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - aR_2 \to R_1}$$

$$\left[\begin{array}{c|c} \overbrace{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}^{I} & \overbrace{\begin{array}{cccc} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{array}}^{A^{-1}} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{X = A^{-1}b} = \begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-a^2 \\ a \\ -1 \\ 0 \end{bmatrix}$$

is the (unique) soln. vector.

4) (15 points) Let the $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where a and b are non-zero real parameters.
For which values of a and b we have

a) $A^3 = -I$

$$A^3 = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{cases} a^3 = -1 \\ b^3 = -1 \end{cases} \qquad \boxed{a = b = -1}$$

b) $A^{-1} = A$

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{cases} \frac{1}{a} = a & \Rightarrow \boxed{a = \pm 1} \\ \frac{1}{b} = b & \Rightarrow \boxed{b = \pm 1} \end{cases}$$

c) A^T is non-singular.

$\det(A^T) = \det A = ab \neq 0$ for all a, b

A^T is nonsingular for all a, b.

5) (15 points) Let A and B be 3×3 matrices such that A is invertible, B is not invertible and $det(A) = 3$. If R_A and R_B are the reduced row-echelon forms of A and B, respectively, then

a) Determine $det(AR_A)$

A is invertible $\underset{\text{means}}{\Rightarrow}$ $R_A = I$, $\det(R_A) = 1$

$$\det(AR_A) = \det(A).\det(R_A) = \det(A).\det(I) = 3.1 = \boxed{3}.$$

b) Determine $det(AR_B)$

R_B has at least one zero row since B is not invertible

so $\det(R_B) = 0$ $\quad \det(A.R_B) = \det(A).\det(R_B) = 3(0) = \boxed{0}$

c) Determine all possible values of $det(R_A + R_B)$.

$$R_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_A = I \qquad R_A + R_B = R_A \Rightarrow \det(R_A + R_B) = \boxed{1.}$$

$$\text{or } R_B = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_A + R_B = \begin{bmatrix} 2 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(R_A + R_B) = \boxed{2}$$

$$\text{or } R_B = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \qquad R_A + R_B = \begin{bmatrix} 2 & * & * \\ 0 & 2 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(R_A + R_B) = \boxed{4}$$