Answer Key for Exercise Set I

1. $x = -\frac{2}{7}, \quad y = -\frac{5}{7}, \quad z = \frac{9}{28}, \quad w = -\frac{5}{14}.$

2. $x_1 = \frac{2}{5}s + t + \frac{1}{5},\ x_2 = \frac{3}{5}s - 2t - \frac{1}{5},\ x_3 = s,\ x_4 = t.$

3. $x_1 = s + 1,\ x_2 = 0,\ x_3 = -s,\ x_4 = s.$

4. There are four cases: i) $a = 0,\ b = 2$, ii) $a = 0,\ b \neq 2$, iii) $a \neq 0,\ b \neq 2$ and iv) $a \neq 0,\ b = 2$.
i) If $a = 0,\ b = 2$ then the reduced row-echelon form of the matrix is:

$$\begin{bmatrix} 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and in this case there is two-parameter solution (since there are 2 parameters (=free variables), namely x_1 and x_2). More precisely, the solution set is $\{x_1 = s,\ x_2 = t,\ x_3 = 1;\ s \in \mathbb{R},\ t \in \mathbb{R}\}$.
ii) If $a = 0,\ b \neq 2$, then the reduced row-echelon form of the matrix is:

$$\begin{bmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which shows that there is no solution in this case.
iii) If $a \neq 0$ and $b \neq 2$ then the row-echelon form of the matrix is:

$$\begin{bmatrix} \mathbf{1} & 0 & \frac{b}{a} & \frac{2}{a} \\ 0 & \mathbf{1} & \frac{4-b}{a} & \frac{2}{a} \\ 0 & 0 & \mathbf{1} & 1 \end{bmatrix}.$$

Which implies there is a unique solution, namely, $x_1 = \frac{2-b}{a}$, $x_2 = \frac{b-2}{a}$, and $x_3 = 1$.
iv) If $a \neq 0$ and $b = 2$ then the reduced row-echelon form of the matrix is:

$$\begin{bmatrix} \mathbf{1} & 0 & \frac{2}{a} & \frac{2}{a} \\ 0 & \mathbf{1} & \frac{2}{a} & \frac{2}{a} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which implies there is one-parameter solution, namely $x_1 = -\frac{2}{a}s + \frac{2}{a}$, $x_2 = -\frac{2}{a} + \frac{2}{a}$, and $x_3 = s$.

5. a) $x_1 = -2s + 15,\ x_2 = -3s + 7,\ x_3 = s.$ **b)** $x_1 = -4s - 2t + 10,\ x_2 = -t + 8,\ x_3 = s,\ x_4 = t.$ **c)** $x_1 = 1,\ x_2 = 0,\ x_3 = -1.$ **d)** No solution.

6. a) $a \in \mathbb{R} \backslash \{-3, 5\}$. **b)** $a = -3$ or $a = 5$.

7. a) $x_1 = \frac{2}{3}a - \frac{1}{9}b,\ x_2 = -\frac{1}{3}a + \frac{2}{9}b.$ **b)** $x_1 = a - \frac{1}{3}c,\ x_2 = a - \frac{1}{2}b,\ x_3 = -a + \frac{1}{2}b + \frac{1}{3}c.$

8. If $a=-4$ then there is no solution. If $a=4$ then there are infinitely many solutions. If $|a|\neq 4$ then there is exactly one solution.

9. $\alpha=\frac{\pi}{2},\ \beta=\pi,\ \gamma=0$ and $\alpha=\frac{\pi}{2},\ \beta=\pi,\ \gamma=\pi.$

10. $\lambda=2,\ \lambda=4.$

11. $x=-\frac{53}{19},\ y=\frac{53}{26},\ z=-\frac{53}{5}.$

12. Suppose that $ad-bc\neq 0$. There are two cases: i) $a\neq 0$ or ii) $a=0$.
i) ($a\neq 0$) Since $a\neq 0$, we can divide Row 1 by a to obtain the first leading **1**:

$$\begin{bmatrix} \mathbf{1} & \frac{b}{a} \\ c & d \end{bmatrix}.$$

Add $-c$ times Row 1 to Row 2:

$$\begin{bmatrix} \mathbf{1} & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

To obtain the second leading **1** divide Row 2 by $\frac{a}{ad-bc}$ (This is possible because $ad-bc\neq 0$):

$$\begin{bmatrix} \mathbf{1} & \frac{b}{a} \\ 0 & \mathbf{1} \end{bmatrix}.$$

Finally add $-\frac{b}{a}$ times Row 2 to Row 1.
ii) ($a=0$) First note that: $b\neq 0$ and $c\neq 0$ (Otherwise $ad-bc$ would be zero). Now, the augmented matrix has the form:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}.$$

Interchange Rows 1 and 2 to obtain a nonzero entry at the top of the leftmost nonzero column:

$$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}.$$

Divide Row 1 by c to obtain the first leading **1**:

$$\begin{bmatrix} \mathbf{1} & \frac{d}{c} \\ 0 & b \end{bmatrix}.$$

Divide Row 2 by b to obtain the second leading **1**:

$$\begin{bmatrix} \mathbf{1} & \frac{d}{c} \\ 0 & \mathbf{1} \end{bmatrix}.$$

Adding $-\frac{d}{c}$ times Row 2 to Row 1 completes the Gauss-Jordan elimination.

13. a) $\begin{bmatrix} \mathbf{1} & 0 & 0 & 6 & 9 & 3 \\ 0 & \mathbf{1} & 0 & 2 & 5 & 2 \\ 0 & 0 & \mathbf{1} & -3 & -6 & -3 \end{bmatrix}.$

b) $x_1 = -6s - 9t + 3,\ x_2 = -2s - 5t + 2,\ x_3 = 3s + 6t - 3,\ x_4 = s,\ x_5 = t.$

14. $\lambda = \frac{1+\sqrt{5}}{2},\ \lambda = \frac{1-\sqrt{5}}{2}.$

15. a) $a = 2$. **b)** $x_1 = \frac{9}{7} - \frac{38}{7},\ x_2 = -\frac{22}{7}s + \frac{124}{7},\ x_3 = -\frac{10}{7} + \frac{22}{7},\ x_4 = s.$

16. a) $\begin{bmatrix} \mathbf{1} & -1 & 0 & 0 & -5 \\ 0 & 0 & \mathbf{1} & 0 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}.$

b) $x_1 = s - 5,\ x_2 = s,\ x_3 = 4,\ x_4 = 0.$

17. a) The reduced row-echelon form of the first matrix is: $\begin{bmatrix} \mathbf{1} & 0 & 18 & 11 & 0 & 0 \\ 0 & \mathbf{1} & -9 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}.$

The second matrix is already in reduced row-echelon form.

b) The first system is not consistent, the second system is consistent.

c) Solution for the second system is: $x_1 = -3s + 5,\ x_2 = s,\ x_3 = 4,\ x_4 = 2.$

18. $x_1 = 0,\ x_2 = 0,\ x_3 = 0.$