Answer Key for Exercise Set II

1. a) $\begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$, **b)** $\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, **c)** $\begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$, **d)** $\begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$, **e)** Undefined,
f) $\begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$, **g)** $\begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$, **h)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, **i)** 5, **j)** -25, **k)** 168, **l)** Undefined.

2. a) $\begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, **b)** $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, **c)** $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, **d)** Undefined, **e)** $\begin{bmatrix} -1/4 & 3/2 \\ 9/4 & 0 \\ 3/4 & 9/4 \end{bmatrix}$,
f) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, **g)** $\begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$, **h)** $\begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$.

3. a) $\begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$, **b)** Undefined, **c)** $\begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$, **d)** $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$, **e)** $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$,
f) $\begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$, **g)** $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$, **h)** $\begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$, **i)** 61, **j)** 35, **k)** 28.

4. a) $\begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$, **b)** Undefined, **c)** $\begin{bmatrix} 2 & -10 & 11 \\ 13 & 2 & 5 \\ 4 & -3 & 13 \end{bmatrix}$, **d)** $\begin{bmatrix} 10 & -6 \\ -14 & 2 \\ -1 & -8 \end{bmatrix}$, **e)** $\begin{bmatrix} 40 & 72 \\ 26 & 42 \end{bmatrix}$,
f) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5. a) $\begin{bmatrix} -1 & 23 & -10 \\ 37 & -13 & 8 \\ 29 & 23 & 41 \end{bmatrix}$, **b)** $\begin{bmatrix} -1 & 23 & -10 \\ 37 & -13 & 8 \\ 29 & 23 & 41 \end{bmatrix}$.

6. a) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, **b)** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$, **d)** $\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$.

7. ($\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$):

$$\begin{aligned} \mathrm{tr}(A+B) &= (A+B)_{11} + (A+B)_{22} + (A+B)_{33} + \cdots + (A+B)_{nn} \\ &= ((A)_{11} + (B)_{11}) + ((A)_{22} + (B)_{22}) + ((A)_{33} + (B)_{33}) + \cdots + ((A)_{nn} + (B)_{nn}) \\ &= ((A)_{11} + (A)_{22} + (A)_{33} + \cdots + (A)_{nn}) + ((B)_{11} + (B)_{22}(B)_{33}) + \cdots (B)_{nn}) = \mathrm{tr}(A) + \mathrm{tr}(B). \end{aligned}$$

8. One; namely,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

9. a) $\pm \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, **b)** Four; namely, $\begin{bmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm 3 \end{bmatrix}$.

10. We have $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix}$,
$(AB)^{-1} = \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$, $B^{-1}A^{-1} = \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$, $ABC = \begin{bmatrix} 70 & 45 \\ 122 & 79 \end{bmatrix}$,

$(ABC)^{-1} = \begin{bmatrix} 79/40 & -9/8 \\ -61/20 & 7/4 \end{bmatrix}$, $C^{-1}B^{-1}A^{-1} = \begin{bmatrix} 79/40 & -9/8 \\ -61/20 & 7/4 \end{bmatrix}$. Hence $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

11. a) $\begin{bmatrix} 5/13 & 1/13 \\ -3/13 & 2/13 \end{bmatrix}$, **b)** $\begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$, **c)** $\begin{bmatrix} -2/5 & 1 \\ -1/5 & 3/5 \end{bmatrix}$, **d)** $\begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$.

12. a) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, **b)** $\begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$, **c)** $\begin{bmatrix} 39 & 13 \\ 26 & 13 \end{bmatrix}$.

13. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

14. $\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^{-x} - e^x) \\ \frac{1}{2}(e^{-x} - e^x) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$.

15. (Let A and B be square matrices such that $AB = 0$. If A is invertible, then $B = 0$): If A is invertible then there exists A^{-1} such that $A^{-1}A = I$. Multiplying both sides of the equality $AB = 0$ by A^{-1} from left, we see that $A^{-1}(AB) = A^{-1}0 \Rightarrow (A^{-1}A)B = 0 \Rightarrow IB = 0 \Rightarrow B = 0$.

16. Yes. (If A is a square matrix and n is a positive integer, then $(A^n)^T = (A^T)^n$):
$(A^n)^T = (A(A^{n-1}))^T = (A^{n-1})^T A^T = (A(A^{n-2}))^T A^T = ((A^{n-2})^T A^T)A^T = (A^{n-2})^T (A^T)^2 = \cdots = (A^2)^T (A^T)^{n-2} = A^T A^T (A^T)^{n-2} = (A^T)^n$.

17. $C = -A^{-1}BA^{-1}$.

18. (If A is invertible and $AB = AC$, then $B = C$): If A is invertible then there exists A^{-1} such that $A^{-1}A = I$. Multiplying both sides of the equality $AB = AC$ by A^{-1} from left, we see that $A^{-1}(AB) = (A^{-1}A)C \Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow IB = IC \Rightarrow B = C$.

19. a), c), d), f).

20. a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, **b)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, **c)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, **d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

21. a) $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$, **b)** $\begin{bmatrix} \sqrt{2}/26 & -(3\sqrt{2})/26 & 0 \\ (4\sqrt{2})/26 & \sqrt{2}/26 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **c)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & -1/7 & 1/7 \end{bmatrix}$,
d) Not invertible, **e)** $\begin{bmatrix} -4/5 & 3/5 & 1/5 & 1/5 \\ 3/2 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 4/5 & 2/5 & -1/5 & -1/5 \end{bmatrix}$.

22. a) $x_1 = 16/3,\ x_2 = -4/3,\ x_3 = -11/3$,
b) $x_1 = -5/3,\ x_2 = 5/3,\ x_3 = 10/3$,
c) $x_1 = 3,\ x_2 = 0,\ x_3 = -4$.

23. a) $b_1 = 2b_2$, **b)** $b_1 = b_2 + b_3$, **c)** No restrictions, **d)** $b_1 = b_3 + b_4,\ b_2 = 2b_3 + b_4$.

24. (The equation $Ax = x$ can be rewritten as $(A - I)x = 0$): $Ix = x \Rightarrow Ax = Ix \Rightarrow Ax - Ix = 0 \Rightarrow (A - I)x = 0$.
a) $x_1 = 0,\ x_2 = 0,\ x_3 = 0$;
b) $x_1 = s/2,\ x_2 = 0,\ x_3 = s$.

25. $X = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$.

26. $a = 11,\ b = -9,\ c = -13.$

27. $a = 2,\ b = -1.$

28. $AB = \begin{bmatrix} 1 & -5 \\ -10 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & -10 \\ -5 & 1 \end{bmatrix} \Rightarrow AB \neq BA.$ We also see that AB is not symmetric, and this is consistent with the statement.

$CD = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}, \quad DC = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow CD = DC.$ We also see that CD is symmetric, and this is consistent with the statement.

29. Since, the product of two symmetric matrices is symmetric if and only if the matrices commute, it is enough to find conditions which makes AB symmetric:

$$AB = \begin{bmatrix} 2a+b & 2b+d \\ a-5b & b-5d \end{bmatrix} \quad \text{is symmetric} \quad \Leftrightarrow 2b+d = a-5b \Leftrightarrow a-d = 7b.$$

30. **a)**, **c)** Yes; **b)**, **d)** No, unless $n = 1$.

31. General solution of the system is $x_1 = \frac{5}{4}s + \frac{1}{4},\ x_2 = -\frac{9}{4}s + \frac{35}{4}$ and $x_3 = s$. We should assign a proper value to s to make $x_1,\ x_2$ and x_3 positive integers. One example is $s = 3$, which makes $x_1 = 4,\ x_2 = 2$ and $x_3 = 3$.

32. $x = \frac{5}{9},\ y = 9,\ z = \frac{1}{3}.$

33. $K = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$

34. $a = 2,\ b = -1,\ c = 1.$

35. $a = 1,\ b = -4,\ c = -5.$

36. $a = -\frac{7}{5},\ b = \frac{4}{5},\ c = \frac{3}{5}.$

37. **a)** $\begin{bmatrix} 3 & 0 & 6 \\ -4 & -7 & 1 \\ -1 & 3 & -2 \end{bmatrix}$, **b)** $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$, **c)** $x_1 = -6,\ x_2 = 5,\ x_3 = -1.$

38. $\begin{bmatrix} 2 & 0 & 1 \\ -3 & 3 & 3 \\ -2 & 0 & 5 \end{bmatrix}.$