

ANSWER KEY FOR EXERCISE SET III

1. **a)** $\lambda = 1$, $\lambda = -3$, **b)** $\lambda = -2$, $\lambda = 3$, $\lambda = 4$.

2. $x = \frac{3 \pm \sqrt{33}}{4}$.

3.

$$\begin{aligned} \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix} &= \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ -\cos \theta & \sin \theta & 1 \end{vmatrix} = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{vmatrix} = \sin^2 \theta + \cos^2 \theta = 1. \end{aligned}$$

4. $AB = \begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix}$, $BA = \begin{bmatrix} ad & bd + ce \\ 0 & cf \end{bmatrix} \Rightarrow AB = BA \Leftrightarrow ae + bf = bd + ce \Leftrightarrow$
 $(bd - bf) - (ea - ec) = b(d - f) - e(a - c) = 0 \Leftrightarrow \begin{vmatrix} b & a - c \\ e & d - f \end{vmatrix} = 0.$

5. **a)** -2 , **b)** 0 , **c)** 0 , **d)** -1 .

6. **a)** 30 , **b)** 5 , **c)** -17 , **d)** 33 , **e)** 39 , **f)** 6 , **g)** $-1/6$, **h)** -2 .

7. **a)** -6 , **b)** 72 , **c)** -6 , **d)** 18 .

8.

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & b^2 - a^2 & c^2 - a^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & (b + a)(b - a) & (c + a)(c - a) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & 0 & -(b + a)(c - a) + (c + a)(c - a) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & 0 & (c - a)((c + a) - (b + a)) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b - a & c - a \\ 0 & 0 & (c - a)(c - b) \end{vmatrix} = (b - a)(c - a)(c - b). \end{aligned}$$

9. $x = 0$, $x = -1$, $x = 1/2$.

10. $x = 1$, $x = -3$.

11. **a)** Invertible, **b)** Not invertible.

12. **a)** -189 , **b)** $-1/7$, **c)** $-8/7$, **d)** $-1/56$, **e)** 7 .

13.

$$\begin{aligned} \begin{vmatrix} b + c & c + a & b + a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} b + c & c + a & b + a \\ a + (b + c) & b + (c + a) & c + (b + a) \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} b + c & c + a & b + a \\ a + b + c & a + b + c & a + b + c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} b + c & c + a & b + a \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0. \end{aligned}$$

14. a) $k = \frac{5 \pm \sqrt{17}}{2}$, b) $k = -1$.

15.

$$\begin{aligned} \begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha + \sin^2 \alpha & \cos^2 \beta + \sin^2 \beta & \cos^2 \gamma + \sin^2 \gamma \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0. \end{aligned}$$

Since $\begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows that $\begin{bmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible.

16. (A square matrix A is invertible if and only if $A^T A$ is invertible): A is invertible $\Leftrightarrow |A| \neq 0 \Leftrightarrow |A^T A| = |A^T| |A| = |A| |A| = |A|^2 \neq 0 \Leftrightarrow A^T A$ is invertible.

17. a) $M_{11} = 29$, $M_{12} = 21$, $M_{13} = 27$, $M_{21} = -11$, $M_{22} = 13$, $M_{23} = -5$, $M_{31} = -19$, $M_{32} = -19$, $M_{33} = 19$.

b) $A_{11} = 29$, $A_{12} = -21$, $A_{13} = 27$, $A_{21} = 11$, $A_{22} = 13$, $A_{23} = 5$, $A_{31} = -19$, $A_{32} = 19$, $A_{33} = 19$.

18. a) $M_{13} = 0$, $A_{13} = 0$, b) $M_{23} = -96$, $A_{23} = 96$, c) $M_{22} = -48$, $A_{22} = -48$, d) $M_{21} = 72$, $A_{21} = -72$.

19. a) $\text{adj}(A) = \begin{bmatrix} 29 & 11 & -19 \\ -21 & 13 & 19 \\ 27 & 5 & 19 \end{bmatrix}$, b) $A^{-1} = \begin{bmatrix} 29/152 & 11/152 & -19/152 \\ -21/152 & 13/152 & 19/152 \\ 27/152 & 5/152 & 19/152 \end{bmatrix}$.

20. a) -40 , b) -66 , c) 0 , d) $k^3 - 8k^2 - 10k + 95$, e) -240 , f) 0 .

21. a) $A^{-1} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$, b) $A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$, c) $A^{-1} = \begin{bmatrix} 1/2 & 3/2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$,

d) $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -4 & 1 & 0 \\ 29/12 & -1/2 & 1/6 \end{bmatrix}$.

22. $A^{-1} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

23. a) $x_1 = 3/11$, $x_2 = 2/11$, $x_3 = -1/11$, b) $x_1 = 5$, $x_2 = 8$, $x_3 = 3$, $x_4 = -1$, c) Cramer's rule does not apply.

24. $y = 0$.

25. -1080 .

26. a) $|A| = 0$, b) No, because of part a), c) 0 .

27. $\lambda = \frac{-1 \pm \sqrt{5}}{2}$.

28. $\lambda = 2$, $\lambda = -3$.

29. 72 .

30. $|A|^{n-2} A$ where n is the number of rows (or columns) of A .

31. $\begin{bmatrix} -3s & s \\ -3t & t \end{bmatrix}$, $s, t \in \mathbb{R}$.

32. a) 8 , b) $1/64$, c) $1/4$, d) $2401/4$.

33. $|A| = 6$, $|B| = 36$, $|B^T| = 36$, $|B^{-1}| = 1/36$.