Answer Key for Exercise Set IV

VECTORS IN 2-SPACE AND 3-SPACE

1. $x - 2y - z = 1$.

2. (a) $x = 1 + t,\ y = -t,\ z = 1 + t,\ t \in \mathbb{R}$.

   (b) $(2, -1, 2)$.

3. Write the equations of lines in the standard form:

$$\ell_1 : \ x = 3 + 4t,\ y = 4 + t,\ z = 1,\ t \in \mathbb{R}, \quad \ell_2 : \ x = -1 + 12s,\ 7 + 6s,\ 5 + 3s,\ s \in \mathbb{R}.$$

   At the intersection point(s) the $z-$coordinate should be 1 (because, the $z-$coordinates of the points on ℓ_1 is always 1). So, find s which makes $z = 1$ on ℓ_2:

$$5 + 3s = 1 \Rightarrow s = -4/3.$$

   Now, calculate the x and y coordinates of the possible intersection point:

$$y = 7 + 6 \cdot \frac{-4}{3} = -1, \quad x = -1 + 12 \cdot \frac{-4}{3} = -17.$$

   So, if there is an intersection point, it should be $(-17, -1, 1)$. We need to check that ℓ_1 passes through $(-17, -1, 1)$. It is clear that, if $t = -5$ then $x = -17$, $y = -1$ and $z = 1$. Thus ℓ_1 and ℓ_2 intersect.

   The equation of the plane containing these two lines, is: $x - 4y + 4z + 9 = 0$.

4. (a) $x + y + z = 1$,

   (b) No.

5. There are infinitely many planes passing through $(1, 1, 1)$ and parallel to the indicated line. Here it is enough to find the equation of only one of these planes. It is clear that the plane $z = 1$ is such a plane.

6. (a) $\vec{u} \cdot \vec{v} = 2,\ \vec{u} \times \vec{v} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$.

   (b) $x - 2y + z + 1 = 0$.

   (c) 0.

7. Homework question.

8. $\vec{PQ} \times \vec{PR} = \begin{pmatrix} 26 \\ 4 \\ -7 \end{pmatrix}$.

   (a) $26x + 4y - 7z = 33$.

   (b) $\frac{741}{2}$.

   (c) $\frac{33}{\sqrt{741}}$.

9. $\vec{v}_1 = \begin{pmatrix} \frac{1+4\sqrt{2}}{3} \\ \frac{8+\sqrt{2}}{3} \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} \frac{2-4\sqrt{2}}{3} \\ \frac{4-\sqrt{2}}{3} \end{pmatrix}$. It is clear that $\vec{v} = \vec{v}_1 + \vec{v}_2$.

10. (a) Since $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$, it follows that $\vec{u}$, $\vec{v}$ and $\vec{w}$ lie in the same plane.

    (b) $\frac{2\pi}{3}$.

11. (a) $\sqrt{10}$.

    (b) 30.

12. $x + z = 5$.

13. $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = (\vec{a}\times\vec{a})+(\vec{a}\times\vec{b})-(\vec{b}\times\vec{a})-(\vec{b}\times\vec{b})$. Note that $\vec{a}\times\vec{a}=\vec{0}$, $\vec{b}\times\vec{b}=\vec{0}$ (cross product of parallel vectors are zero), and $\vec{b}\times\vec{a}=-\vec{a}\times\vec{b}$. Hence, $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = \vec{0}+(\vec{a}\times\vec{b})-(-\vec{a}\times\vec{b})-\vec{0} = 2(\vec{a}\times\vec{b})$.

14. (a) Not parallel.
    (b) Parallel.
    (c) Parallel.

15. (a) Parallel.
    (b) Not parallel.

16. (a) Not perpendicular.
    (b) Perpendicular.

17. (a) Perpendicular.
    (b) Not perpendicular.

18. $7x+4y-2z=0$.

19. $\left(-\frac{173}{3}, -\frac{43}{3}, \frac{49}{3}\right)$.

20. Homework question.

21. $x+5y+3z-18=0$.

22. It is easy to show that the points $P_1=(a,0,0)$, $P_2=(0,b,0)$ and $P_3=(0,0,c)$ lie on the plane $M: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (Here, $a\neq 0,\ b\neq 0,\ c\neq 0$):

    P_1 lies on M: $\frac{a}{a}+\frac{0}{b}+\frac{0}{c}=\frac{a}{a}=1$,

    P_2 lies on M: $\frac{0}{a}+\frac{b}{b}+\frac{0}{c}=\frac{b}{b}=1$,

    P_3 lies on M: $\frac{0}{a}+\frac{0}{b}+\frac{c}{c}=\frac{c}{c}=1$.

    Since a plane is determined uniquely by three points not lying on the same line, the plane whose intercepts with the coordinate axes are $x=a$, $y=b$, and $z=$ has the equation should be M.

EUCLIDEAN VECTOR SPACES

1. (a) $[T]=\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$.
   (b) There is no such a vector.

2. Yes, T is one-to-one.

3. (a) Yes, T is one-to-one.
   (b) $T^{-1}((x,y,z)) = (-2x+4y-3z, 6x-11y+9z, 7x-12y+10z)$.

4. In class, we have proved the following:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2. \tag{1}$$

   (a) If $\|\vec{u}+\vec{v}\| = \|\vec{u}-\vec{v}\|$, then it follows from (1) that $\vec{u}\cdot\vec{v}=0$, and hence $\vec{u}$ and $\vec{v}$ are orthogonal.
   (b) Since $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2$ and $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2-2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2$ (see the Exercise 6, in the lecture notes of Chapter 4), it follows that $\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2+2\|\vec{v}\|^2$.

5. (a) $[T]=\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.
   (b) No.

6. (a) Since, $[T]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has determinant $|[T]|=1$, it is invertible and so T s one-to-one.

(b) $P(1, 0, 2, -2)$.

7. We will prove the assertion in two parts:

   i) If T is one-to-one, then $\{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = 0\} = \{0\}$:

   First, let us show that $T(\vec{0}) = \vec{0}$. Note that, $\vec{0} = \vec{0} + \vec{0}$. This implies with linearity property of T that $T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$ and and hence $T(\vec{0}) = 2T(\vec{0})$, which is possible only when $T(\vec{0}) = \vec{0}$. Since T is one-to-one, $\vec{0}$ is the only vector in $\mathbb{R}^n$ such that $T(\vec{0}) = \vec{0}$ and so, $\{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = 0\} = \{0\}$.

   ii) If $\{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = 0\} = \{0\}$, then T is one-to-one:

   Now, suppose that $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$ such that $T(\vec{u}) = T(\vec{v})$. We want to show that $\vec{u} = \vec{v}$.

   Now, let $\vec{w} = \vec{u} - \vec{v}$ and consider $T(\vec{w})$. $T(\vec{w}) = T(\vec{u}) - T(\vec{v})$. Since $T(\vec{u}) = T(\vec{v})$ it follows that $T(\vec{w}) = \vec{0}$. From the assumption $\{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = 0\} = \{0\}$, we know that $\vec{0}$ is the only vector whose image is $\vec{0}$. Thus, $\vec{w} = \vec{0}$ and hence $\vec{u} = \vec{v}$. This proves that T is one-to-one.

8. (a) $[T_4(T_3(T_2(T_1)))] = \begin{bmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{bmatrix}$.

   (b) Yes, because its standard matrix is invertible.

9.

$$[ST] = \begin{bmatrix} | & | & \cdot & | \\ ST(\vec{e}_1) & ST(\vec{e}_2) & \cdot & ST(\vec{e}_n) \\ | & | & \cdot & | \end{bmatrix}.$$

   Note that $ST(\vec{e_1}) = S(T(\vec{e_1})) = [S]T(\vec{e_1})$. So,

$$[ST] = [S] \begin{bmatrix} | & | & \cdot & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdot & T(\vec{e}_n) \\ | & | & \cdot & | \end{bmatrix} = [S][T].$$

10. (a) the reflection about the $x-$axis.

    (b) the clockwise rotation through an angle of $\pi/4$ in $\mathbb{R}^2$.

    (c) multiplication by $\frac{1}{3}$.

    (d) the reflection about the $yz-$plane in $\mathbb{R}^3$.

11. (a) T is a linear transformation.

    (b) T is not a linear transformation.

12. Proved in Exercise 7.

13. It is enough to check that whether $\{\vec{v} \in \mathbb{R}^n : A\vec{v} = 0\} = \{0\}$ or not. (see, Exercise 7). In other another words, whether the homogeneous system $A\vec{v} = \vec{0}$ has unique solution (the trivial solution) or not. Check by yourselves that

    (a) Invertible.

    (b) Not invertible.

14. Consider the vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ Cauchy-Schwarz formula says that:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|. \qquad (2)$$

    Note that $\vec{u} \cdot \vec{v} = a\cos\theta + b\cos\theta$, $\|\vec{u}\| = \sqrt{a^2 + b^2}$ and $\|\vec{v}\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$. Putting these values in (2), we get

$$|a\cos\theta + b\cos\theta| \leq \sqrt{a^2 + b^2}.$$

    Taking squares of the both sides proves the assertion.

15. (a) $\sqrt{2}$.

    (b)