Answer Key for Exercise Set V

## EIGENVALUES AND EIGENVECTORS

1. Eigenvalues of $A$ are 1 and 2. Eigenvectors corresponding to 1, are $t\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $t \in \mathbb{R}$, and eigenvectors corresponding to 2, are $t\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$. Hence, Eigenvalues of $A^{20}$ are 1 and $2^{20}$. Eigenvectors corresponding to 1, are $t\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $t \in \mathbb{R}$, and eigenvectors corresponding to $2^{20}$, are $t\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$.

2. Since $\lambda_1$ is an eigenvalue of $A$, and $\vec{v}$ is a corresponding eigenvector; and $\lambda_2$ is an eigenvalue of $B$, $\vec{v}$ is a corresponding eigenvector, we have $A\vec{v} = \lambda_1\vec{v}$ and $B\vec{v} = \lambda_2\vec{v}$. Then:
   (a) $(AB)\vec{v} = A(B\vec{v}) = A(\lambda_2\vec{v}) = \lambda_2(A\vec{v}) = \lambda_2(\lambda_1\vec{v}) = (\lambda_1\lambda_2)\vec{v}$. This shows that $\lambda_1\lambda_2$ is an eigenvalue, and $\vec{v}$ is a corresponding eigenvector.
   (b) $(A^5 + B^3)\vec{v} = A^5\vec{v} + B^3\vec{v} = \lambda_1^5\vec{v} + \lambda_2^3\vec{v} = (\lambda_1^5 + \lambda_2^3)\vec{v}$. This shows that $\lambda_1^5 + \lambda_2^3$ is an eigenvalue, and $\vec{v}$ is a corresponding eigenvector.

3. (a) Eigenvalues of $A$ are 1, $-3$ and 2. Eigenvalues of $A^3$ are 1, $-27$ and 8.
   (b) No, because 5 is not an eigenvalue of $A$.
   (c) See Exercise 10.
   (d) Eigenvalues of $A + 7I$ are 8, 4 and 9.

4. (a) Eigenvalues of $A$ are 0, $-1$, and the corresponding eigenvectors are:
   $t\begin{bmatrix}1\\-5/2\end{bmatrix}$, $t \in \mathbb{R}$, and $t\begin{bmatrix}1\\-2\end{bmatrix}$, $t \in \mathbb{R}$,
   (b) $P = \begin{bmatrix}1 & 1\\-5/2 & -2\end{bmatrix}$ diagonalizes $A$.
   (c) $\begin{bmatrix}-5 & -2\\10 & 4\end{bmatrix}$.

5. (a) $\lambda^2 - 2\lambda - 3 = 0$.
   (b) $\lambda^2 - 8\lambda + 16 = 0$.

6. (a) Eigenvalues are 3, $-1$, and the corresponding eigenvectors are:
   $t\begin{bmatrix}1/2\\1\end{bmatrix}$, $t \in \mathbb{R}$, and $t\begin{bmatrix}0\\1\end{bmatrix}$, $t \in \mathbb{R}$.
   (b) Eigenvalue is 4, and the corresponding eigenvectors are:
   $t\begin{bmatrix}3/2\\1\end{bmatrix}$, $t \in \mathbb{R}$.

7. (a) Eigenvalues are 1, 2, 3, and the corresponding eigenvectors are:
   $t\begin{bmatrix}0\\1\\0\end{bmatrix}$, $t \in \mathbb{R}$, $t\begin{bmatrix}-1/2\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$, and $t\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$.
   (b) Eigenvalues are 0, $\sqrt{2}$, $-\sqrt{2}$, and the corresponding eigenvectors are:
   $t\begin{bmatrix}5/3\\1/3\\1\end{bmatrix}$, $t \in \mathbb{R}$, $t\begin{bmatrix}1/7(15+5\sqrt{2})\\1/7(-1+2\sqrt{2})\\1\end{bmatrix}$, $t \in \mathbb{R}$, and $t\begin{bmatrix}1/7(15-5\sqrt{2})\\1/7(-1-2\sqrt{2})\\1\end{bmatrix}$, $t \in \mathbb{R}$.
   (c) Eigenvalue is $-8$, and the corresponding eigenvectors are:
   $t\begin{bmatrix}-1/6\\-1/6\\1\end{bmatrix}$, $t \in \mathbb{R}$.

(d) Eigenvalue is 2, and the corresponding eigenvectors are:
$$t\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

(e) Eigenvalue is 2, and the corresponding eigenvectors are:
$$t\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

(f) Eigenvalues are $-4,\ 3$, and the corresponding eigenvectors are:
$$t\begin{bmatrix} -2 \\ 8/3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \text{ and } t\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \ t \in \mathbb{R}.$$

8. Eigenvalues of $A^{25}$ are $1,\ -1$, and the corresponding eigenvectors are:
$$t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ t,\ s \in \mathbb{R}, \text{ and } t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \ t \in \mathbb{R}.$$

9. Since $\lambda$ is an eigenvalue of $A$ and $\vec{v}$ is a corresponding eigenvector, we have $A\vec{v} = \lambda\vec{v}$. Multiplying the both sides of the equation by $A^{-1}$ from left, we see that $\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$. Dividing both sides of the equation by $\lambda$ (think why, division by $\lambda$ is possible, that is why $\lambda \neq 0$), we get $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$, and this shows that $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$ and $\vec{v}$ is a corresponding eigenvector.

10. Since $\lambda$ is an eigenvalue of $A$ and $\vec{v}$ is a corresponding eigenvector, we have $A\vec{v} = \lambda\vec{v}$. Note that, $(A - sI)\vec{v} = A\vec{v} - sI\vec{v} = \lambda\vec{v} - s\vec{v} = (\lambda - s)\vec{v}$. Then, it is clear that $\lambda - s$ is an eigenvalue of $A - sI$, and $\vec{v}$ is a corresponding eigenvector.

11. (a) Eigenvalues of $A^{-1}$ are $1,\ 1/2,\ 1/3$, and the corresponding eigenvectors are:
$$t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \quad t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \ t \in \mathbb{R}, \text{ and } t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

(b) Eigenvalues of $A - 3A$ are $-2,\ -1,\ 0$, and the corresponding eigenvectors are:
$$t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \quad t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \ t \in \mathbb{R}, \text{ and } t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

(c) Eigenvalues of $A + 2I$ are $3,\ 4,\ 5$, and the corresponding eigenvectors are:
$$t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \quad t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \ t \in \mathbb{R}, \text{ and } t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

12. Eigenvalues of $A^T$ are the roots of the polynomial $|A^T - \lambda I|$. Since $|B^T| = |B|$ for any square matrix $B$, the same is true for $A^T - \lambda I$. Thus $|A^T - \lambda I| = |(A^T - \lambda I)^T| = |(A^T)^T - (\lambda I)^T|$. Since $(A^T)^T = A$ and $(\lambda I)^T = \lambda I$, we have $|A^T - \lambda I| = |(A^T)^T - (\lambda I)^T| = |A - \lambda I|$. Hence the polynomials $|A^T - \lambda I|$ and $|A - \lambda I|$ are same. So, their roots are same, that is $A^T$ and $A$ has the same eigenvalues.

13. (a) Not diagonalizable.

(b) Not diagonalizable.

14. $P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

15. $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$.

17. $\begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

18. Homework question.